Lecture #1: Partial Recap of ODEs.

In this course we often need to use techniques from ordinary differential equations. Here I want to review some common ODEs that will come up in the course.

Example:

rate of change w.r.t time

$$\begin{cases} \dfrac{dx}{dt} = e^{-t} \leftarrow \text{function of time}, & t>0 \\ \underbrace{x(0)=x_0}_{\text{intial condition}} \end{cases}$$

You can antidifferentiate, but I want to be a little more careful.

$$\Rightarrow \int_0^t \frac{dx}{ds}\,ds = \int_0^t e^{-s}\,ds$$

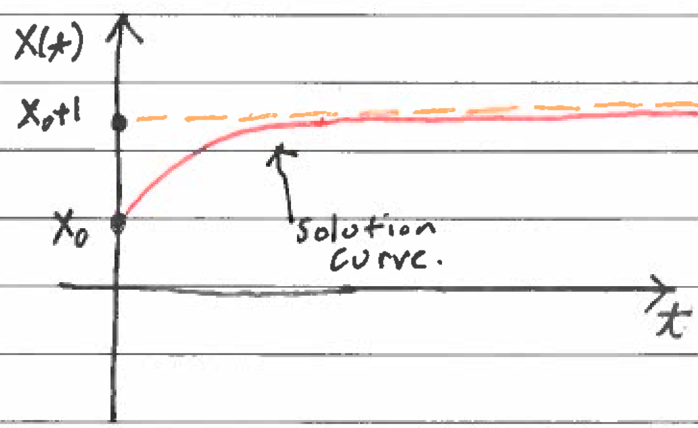
* s is a dummy variable of integration *

Let $u = x(s)$

- $du = \dfrac{dx}{ds}\,ds$
- $u(0) = x(0) = x_0$
- $u(t) = x(t)$

Therefore,

$$\int_{x_0}^{x(t)} du = \int_0^t e^{-s}\,ds$$



$$\Rightarrow x(t) - x_0 = 1 - e^{-t}$$

$$\Rightarrow x(t) = x_0 + 1 - e^{-t}$$

In shorthand notation, we would write:

$$\frac{dx}{dt} = e^{-t}, \quad x(0) = x_0$$

$$\Rightarrow dx = e^{-t} dt$$

$$\Rightarrow \int_{x_0}^{x} dx = \int_0^t e^{-t} dt \Rightarrow x = x_0 + 1 - e^{-t}$$

This is fine, but be careful!!

- $\frac{dx}{dt}$ is not a fraction. This matters when we start talking about partial derivatives.
- I strongly recommend using dummy variables. This matters when we start solving PDEs with multiple variables.

Example:

$$\frac{dx}{dt} = \frac{1+x}{1+t^2}, \quad x(0) = x_0$$ *This is called a seperable ODE*

$$\Rightarrow \frac{1}{1+x}\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\Rightarrow \int_0^t \frac{1}{1+x(s)}\frac{dx}{ds} ds = \int_0^t \frac{1}{1+s^2} ds$$

Let $u = x(s)$

- $du = \frac{dx}{ds} ds$
- $u(0) = x(0) = x_0$
- $u(t) = x(t)$

$$\Rightarrow \int_{x_0}^{x(t)} \frac{1}{1+u} du = \int_0^t \frac{1}{1+s^2} ds$$

$$\Rightarrow \ln(1+x) - \ln(1+x_0) = \tan^{-1}(t)$$

$$\Rightarrow \ln(1+x) = \ln(1+x_0) + \tan^{-1}(t)$$

$$\Rightarrow x = (1+x_0)e^{\tan^{-1}(t)} - 1$$

In the previous examples, we were using the Fundamental Theorem of Calculus on the right hand side and the reverse of the chain rule on the left hand side to undo differentiation. In the next example, we reverse the product rule.

Example:

$$\frac{dy}{dx} + 2xy = x, \quad y(0) = 3$$

Idea: multiply both sides by $e^{f(x)}$

$$\Rightarrow e^{f(x)}\frac{dy}{dx} + 2xe^{f(x)}y = xe^{f(x)}$$

Now, if

$$\frac{d}{dx}\left(e^{f(x)}y\right) = f'(x)e^{f(x)}y + e^{f(x)}\frac{dy}{dx} = e^{f(x)}\frac{dy}{dx} + 2xe^{f(x)}y$$

then

$$f'(x) = 2x \Rightarrow f(x) = x^2.$$

Setting $f(x) = x^2$ we have that

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2}y\right) = xe^{x^2}$$

$$\Rightarrow \int_0^x \frac{d}{ds}\left(e^{s^2}y\right)ds = \int_0^x se^{s^2}ds$$

$$\Rightarrow e^{x^2}y(x) - 3 = \frac{1}{2}e^{x^2} - \frac{1}{2}$$

$$\Rightarrow y(x) = \frac{1}{2} + \frac{5}{2}e^{-x^2}$$

The term e^{x^2} is called an <u>integrating factor</u>

Example:

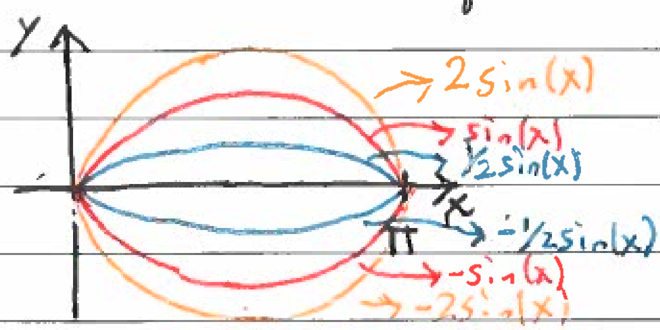
Solve

$$\frac{d^2y^2}{dx^2} = -y,$$

$$\underbrace{y(0)=0,\ y(\pi)=0}_{\text{Boundary conditions}}$$

$y(x) = a\sin(x)$, $a \in \mathbb{R}$, works.

Solutions are not unique!



Example:

Plot solution curves for the ODE

$$\frac{dx}{dt} = f(x) \leftarrow \text{velocity as function of } x$$

$$x(0) = x_0$$

We could try to solve analytically

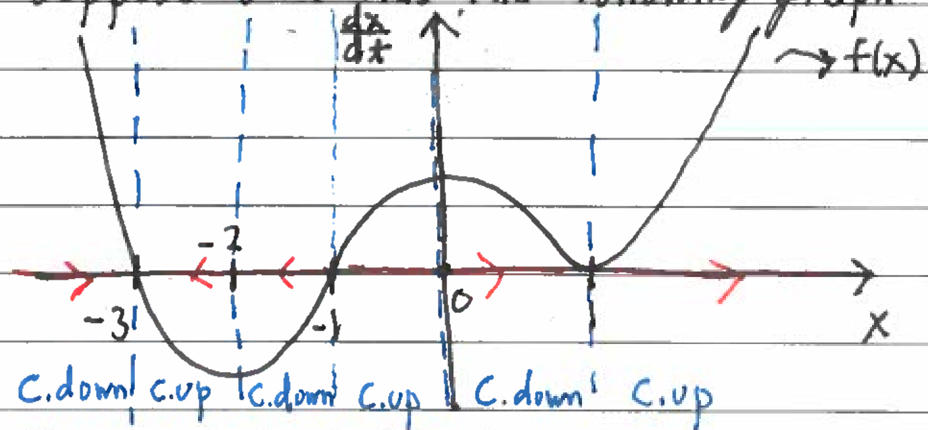
$$t = F(x) - F(x_0) = \int_{x_0}^{x} f(u)\,du$$

$$\Rightarrow x = F^{-1}(t + F(x_0))$$

i) We may not be able to do the integration.

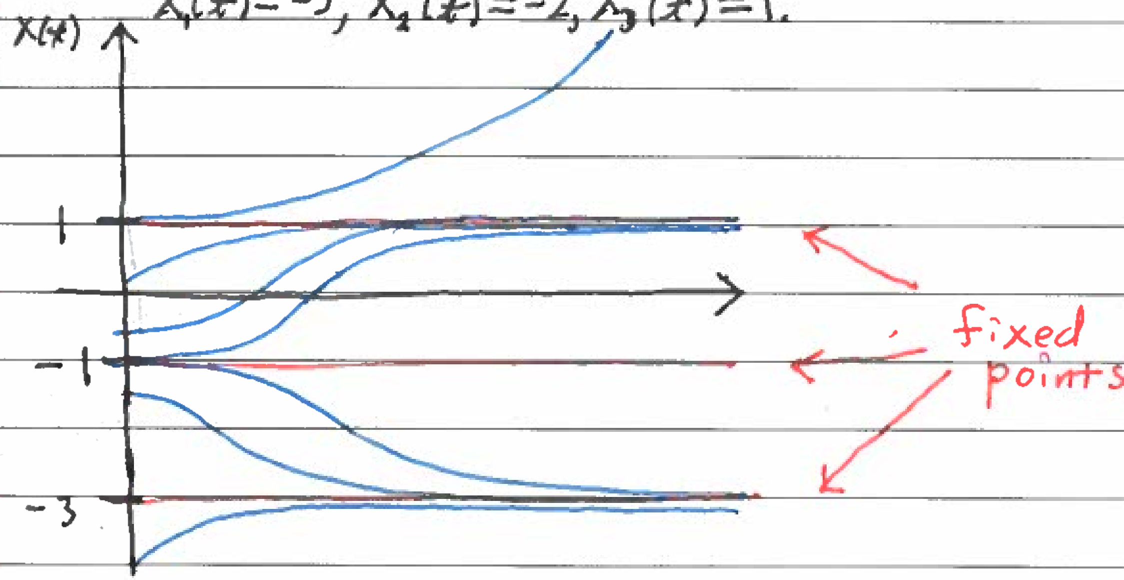
ii) Inverting F could be really hard.

Suppose $f(x)$ has the following graph



Since $f(-3), f(-2), f(1) = 0$ there are three explicit solutions:

$x_1(t) = -3,\ x_2(t) = -2,\ x_3(t) = 1.$



i) $f(x) > 0$, $\frac{dx}{dt} > 0$, x increases in time

ii) $f(x) < 0$, $\frac{dx}{dt} < 0$, x decreases in time

iii) $f(x^*) = 0$, $\frac{dx}{dt} = 0$, x^* is a solution of $\frac{dx}{dt} = f(x)$.

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt} f(x) = f'(x)\frac{dx}{dt} = f'(x) f(x)$$

We can predict the behavior of solutions without solving the ODE!

Theorem - Consider the initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0$$

If $f'(x)$ is continuous then the initial value problem has a solution on some interval $(-\tau, \tau)$ about $t=0$, and the solution is unique.