Lecture #10: Convergence of Functions

Three notions of convergence we are interested in:

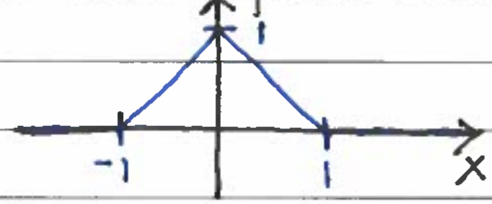
1. We say $f_n \to f$ pointwise on $[a,b]$ if for all $x \in [a,b]$,
$$\lim_{n\to\infty} f_n(x) = f(x).$$

2. We say $f_n \to f$ in L^2, denoted $f_n \xrightarrow{L^2} f$, on $[a,b]$ if
$$\lim_{n\to\infty} \int_a^b (f_n(x)-f(x))^2\,dx = 0$$

3. We say $f_n \to f$ uniformly, denoted $f_n \xrightarrow{L^\infty} f$, on $[a,b]$ if
$$\lim_{n\to\infty} \max_{a\le x\le b} |f_n(x)-f(x)| = 0.$$

Example:

Suppose u satisfies
$$u_{tt} = c^2 u_{xx},$$
$$u(x,0) = f(x),$$
where $f(x)$ is plotted below:



Therefore, $u(x,t) = \frac{1}{2}(f(x-ct)+f(x+ct))$.

1. $\lim_{t\to\infty} u(x,t) = 0 \Rightarrow u(x,t) \to 0$ pointwise.

2. $$\lim_{t\to\infty} \int_{-\infty}^{\infty} (u(x,t)-0)^2\,dx = \lim_{t\to\infty} \frac{1}{4}\int_{-\infty}^{\infty} (f(x-ct)^2 + 2f(x-ct)f(x+ct) + f(x+ct)^2)\,dx$$
$$= \lim_{t\to\infty} \frac{1}{4}\int_{-\infty}^{\infty} (f(x-ct)^2 + f(x+ct)^2)\,dx$$
$$= \lim_{t\to\infty} \frac{1}{2}\int_{-\infty}^{\infty} f(x)^2\,dx$$
$$= \lim_{t\to\infty} \int_0^1 x^2\,dx = 1/3$$

$\Rightarrow f_n \not\xrightarrow{L^2} 0.$

3. $\lim_{t\to\infty} \max_{-\infty<x<\infty} |u(x,t)| = \lim_{t\to\infty} \frac{1}{2} = \frac{1}{2}$.

Therefore, $f_n \overset{L^\infty}{\not\to} 0$.

Theorem - If $f_n \xrightarrow{L^\infty} f$ on $[a,b]$ then $f_n \xrightarrow{L^2} f$ on $[a,b]$.

~~proof~~:

$$\int_a^b (f_n(x)-f(x))^2 dx \le \int_a^b \max_{a\le x\le b} (f_n(x)-f(x))^2 dx = (b-a) \max_{a\le x\le b} (f_n(x)-f_n(b))^2$$

By the squeeze theorem

$$\lim_{n\to\infty} \int_a^b (f_n(x)-f(x))^2 dx = 0 \Rightarrow f_n \xrightarrow{L^2} f.$$

Example:

Let $f_n(x) = x^n$, for $0 \le x \le 1$. Therefore,

1. $$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$$

2. $\int_0^1 f_n(x)^2 dx = \int_0^1 x^{2n} dx = \frac{1}{2n}$.

Consequently,

$$\lim_{n\to\infty} \int_0^1 f_n(x)^2 dx = 0$$

$$\Rightarrow f_n \xrightarrow{L^2} 0.$$

3. $\max_{0\le x\le 1} |f_n(x)| = 1$

$$\Rightarrow \lim_{n\to\infty} \max_{0\le x\le 1} |f_n(x)| = 1.$$

Therefore,

$$f_n \overset{L^\infty}{\not\to} 0.$$