Lecture #11: Orthogonal Functions

Definition - The space of functions $L^2([a,b])$ consists of real valued functions f defined on $[a,b]$ for which
$$\int_a^b f(x)^2 dx < \infty$$

Definition - The inner product on $L^2([a,b])$ is defined by $\langle f,g\rangle = \int_a^b f(x)g(x)dx.$

Definition - The norm on $L^2([a,b])$ is defined by
$$\|f\| = \langle f,f\rangle^{1/2} = \left(\int_a^b f(x)^2 dx\right)^{1/2}$$

Definition - The angle θ between two functions is defined by
$$\langle f,g\rangle = \|f\|\cdot\|g\| \cos\theta.$$
Two functions are orthogonal if
$$\langle f,g\rangle = 0.$$

Examples:

The functions $\{\sin(x), \sin(2x), \ldots\}$ form an orthogonal system on $[0,\pi]$ since
$$\int_0^\pi \sin(nx)\sin(mx)dx = \int_0^\pi \frac{e^{inx}-e^{-inx}}{2i}\cdot\frac{e^{imx}-e^{-imx}}{2i}dx$$
$$= \int_0^\pi \left(\frac{e^{i(n+m)x}+e^{-i(n+m)x}}{-4} - \frac{e^{i(n-m)x}+e^{-i(n-m)x}}{-4}\right)dx$$
$$= \frac{1}{2}\int_0^\pi \cos((n-m)x) - \cos((n+m)x)dx$$
$$= \begin{cases} \pi/2 & \text{if } n=m \\ 0 & n\neq m \end{cases}$$

Example:

The point of an orthogonal system is we can expand a function in terms of these coefficients. Assume that on $[0,\pi]$ we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

where f is defined by

$$f(x) = \begin{cases} 0, & x < \pi/4,\ x > 3\pi/4 \\ 1, & \pi/4 \leq x \leq 3\pi/4 \end{cases}$$

To determine coefficients we use orthogonality

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \langle \sin(mx), f(x) \rangle = \sum_{n=1}^{\infty} b_n \langle \sin(mx), \sin(nx) \rangle$$

$$= b_m \cdot \pi/2$$

$$\Rightarrow b_m = \frac{2}{\pi} \int_0^{\pi} \sin(mx) f(x)\,dx$$

$$= 2/\pi m \int_{\pi/4}^{3\pi/4} \sin(mx)\,dx$$

$$= \frac{2}{\pi m}\left(\cos\left(\frac{m\pi}{4}\right) - \cos\left(\frac{3m\pi}{4}\right)\right)$$

Therefore,

$$f(x) = \frac{2\sqrt{2}}{\pi}\left(\sin(x) - \frac{\sin(3x)}{3} - \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \frac{\sin(9x)}{9} + \dots \right)$$

Substituting in $\pi/2$ we obtain:

$$1 = \frac{2\sqrt{2}}{\pi}\left(1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \dots \right)$$

$$\Rightarrow 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \dots = \frac{\pi}{2\sqrt{2}}.$$

Definition - An orthonormal system $\{f_n\}$ in $L^2([a,b])$ is complete if and only if there is no nontrivial $f \in L^2([a,b])$ which is orthogonal to all f_n.

Theorem (HUGE): If $\{f_1, f_2, \dots\}$ is complete than any function $f \in L^2([a,b])$ can be expanded in a Fourier Series

$$f(x) \sim \sum_{n=1}^{\infty} a_n f_n(x).$$

Moreover, if $S_N(x) = \sum_{n=1}^{N} a_n f_n(x)$ then

$$S_N \xrightarrow{L^2} f.$$

If f is continuous at x_0 then

$$\lim_{N \to \infty} S_N(x_0) = f(x_0).$$

Best Approximation:

Let $\{c_n\}$ be the Fourier coefficients for a function $f \in L^2([a,b])$. If a_n is any other sequence

$$\|f - \sum_{n=1}^{N} c_n f_n\|^2 \le \|f - \sum_{n=1}^{N} a_n f_n\|^2,$$

assuming f_n are orthonormal.

proof:

$$\begin{aligned}
\|f - \sum a_n f_n\|^2 &= \langle f - \sum a_n f_n, f - \sum a_n f_n \rangle \\
&= \langle f, f \rangle - 2\langle \sum a_n f_n, f \rangle + \langle \sum a_n f_n, \sum a_n f_n \rangle \\
&= \|f\|^2 - 2\sum a_n c_n + \sum a_n^2 \\
&= \|f\|^2 - 2\sum a_n c_n + \sum c_n^2 - \sum c_n^2 + \sum a_n^2 \\
&= \|f\|^2 - \sum c_n^2 + \sum (a_n - c_n)^2 \\
&= \|f - \sum c_n f_n\|^2 + \sum (a_n - c_n)^2
\end{aligned}$$

$$\Rightarrow \|f - \sum c_n f_n\|^2 \le \|f - \sum a_n f_n\|^2$$

Corollary (Bessel's Inequality):

$$\sum_{n=1}^{N} c_n^2 \leq \|f\|^2$$

Theorem (Parseval's Equality):

$\{f_1, f_2, \ldots\}$ is a complete orthonormal system if and only if for each $f \in L^2$ with Fourier coefficients $\{c_n\}$

$$\sum_{n=1}^{\infty} c_n^2 = \|f\|^2$$