Lecture #12: Self Adjoint Operators

Definition - Let $V \subseteq L^2([a,b])$. An operator $\mathcal{L}$ on V is called self-adjoint if for all $f, g \in V$, $\langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}g\rangle$.

Example - Let $V = \{f \in L^2([a,b]) : f(a) = f(b) = 0\}$ and $\mathcal{L} = \frac{d^2}{dx^2}$. Therefore, for $f, g \in V$ we have that

$$\begin{aligned}
\langle \mathcal{L}f, g\rangle &= \int_a^b f''(x)g(x)\,dx \\
&= f'(x)g(x)\Big|_a^b - \int_a^b f'(x)g'(x)\,dx \\
&= -\int_a^b f'(x)g'(x)\,dx \\
&= +f(x)g'(x)\Big|_a^b + \int_a^b f(x)g''(x)\,dx \\
&= \int_a^b f(x)g''(x)\,dx \\
&= \langle f, \mathcal{L}g\rangle.
\end{aligned}$$

Consequently, on V $\mathcal{L}$ is self adjoint.

Theorem - A self-adjoint operator only has real eigenvalues.

proof:

Suppose $\mathcal{L}f = \lambda f$ and $\lambda \in \mathbb{C}$ therefore there exists g such that $\mathcal{L}g = \bar{\lambda} g$ and thus

$$\langle \mathcal{L}f, g\rangle = \langle \lambda f, g\rangle = \lambda\langle f, g\rangle$$

$$\langle \mathcal{L}f, \bar{g}\rangle = \langle f, \mathcal{L}\bar{g}\rangle = \langle f, \bar{\lambda} g\rangle = \bar{\lambda}\langle f, g\rangle$$

Consequently $\lambda\langle f, g\rangle = \bar{\lambda}\langle f, \bar{g}\rangle \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

$\blacksquare$

Theorem - The eigenfunctions of a self-adjoint operator are orthogonal

proof:

Let f, g be eigenfunctions with eigenvalues $\lambda, \nu \in \mathbb{R}$. Therefore,

$$\lambda\langle f, g\rangle = \langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}g\rangle = \nu\langle f, g\rangle$$

$$\Rightarrow (\lambda - \nu)\langle f, g\rangle = 0.$$

$$\Rightarrow \langle f, g\rangle = 0.$$

$\blacksquare$

Example:

Let $V = \{f \in L^2([-1,1]) : f(-1) = f(1), f'(-1) = f'(1)\}$ and $\mathcal{L} = \frac{d^2}{dx^2}$. Therefore, for $f, g \in V$ we have

$$
\begin{aligned}
\langle \mathcal{L}f, g\rangle &= \int_{-1}^{1} f''(x) g(x)\,dx \\
&= f'(x)g(x)\Big|_{-1}^{1} - \int_{-1}^{1} f'(x) g'(x)\,dx \\
&= \cancelto{0}{f'(1)g(1) - f'(-1)g(-1)} - \cancelto{0}{f(x)g'(x)\Big|_{-1}^{1}} + \int_{-1}^{1} f(x) g''(x)\,dx \\
&= \langle f, \mathcal{L}g\rangle .
\end{aligned}
$$

Eigenvalues λ of $\mathcal{L}$ with corresponding eigenvectors satisfy

$$
\begin{aligned}
&\mathcal{L}f = \lambda f \\
&\Rightarrow f''(x) = \lambda f \\
&\Rightarrow f(x) = A\cos(\sqrt{-\lambda}\,x) + B\sin(\sqrt{-\lambda}\,x).
\end{aligned}
$$

Since $\lambda \in \mathbb{R}$ it follows that $\lambda \leq 0$. To ensure the functions are in V we must have that

$$
\begin{aligned}
&\sqrt{-\lambda} = 2\pi n,\ n \in \mathbb{N} \quad \text{or} \quad \sqrt{-\lambda} = 0. \\
&\Rightarrow \lambda = -4\pi^2 n^2 \quad \text{or} \quad \lambda = 0.
\end{aligned}
$$

The set of eigenfunctions are

$$\{1, \cos(2\pi x), \cos(4\pi x), \ldots, \sin(2\pi x), \sin(4\pi x), \ldots\}.$$

Theorem:

If $\mathcal{L}$ is a self adjoint operator on V, then its eigenfunctions form an orthogonal system.