Lecture #14: Heat Equation on a Bounded Domain II

Von-Neumann Boundary Conditions

$u_t = u_{xx}$

$u_x(0,t) = u_x(L,t) = 0$ (Insulated Boundary Conditions)

$u(x,0) = f(x)$

Heat is conserved:

Let $H(t) = \int_0^L u(x,t)\,dx$. Therefore,

$$\frac{dH}{dt} = \int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2}\,dx = \left.\frac{\partial u}{\partial x}\right|_0^L = 0.$$

Conjecture,

$$\lim_{t\to\infty} u(x,t) = u^*(x)$$

which is called the steady state solution. A steady state solution does not depend on time and thus

$$\frac{d^2 u^*}{dx^2} = 0$$

$$\Rightarrow u^*(x) = ax + b$$

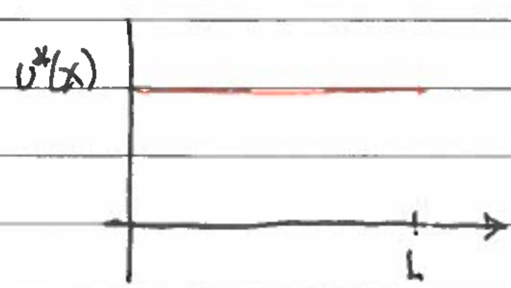
Boundary conditions imply $u^*_x(0) = u^*_x(L) = 0 \Rightarrow a = 0$.

Conservation of heat implies

$$H(0) = \int_0^L u^*(x)\,dx = Lb$$

$$\Rightarrow b = \frac{1}{L}\int_0^L f(x)\,dx$$

$$\boxed{u^*(x) = \frac{1}{L}\int_0^L f(x)\,dx} \longrightarrow \text{Average of initial heat.}$$

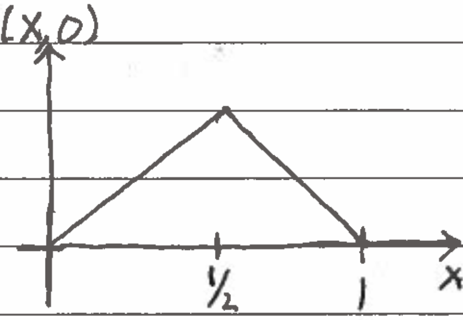


Example:

$u_t = u_{xx}$

$u_x(0,t) = u_x(1,t) = 0$

$u(x,0) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1 \end{cases}$



Let $u(x,t) = X(x)T(t)$. Therefore,

$$X \cdot T' = X'' T$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow T = Ce^{-\lambda t}, \quad X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

Applying boundary conditions:

$$u_x(x,t) = X' \cdot T$$

$$\Rightarrow u_x(0,t) = X'(0)T \Rightarrow X'(0) = 0$$

$$u_x(1,t) = X'(1)T \Rightarrow X'(1) = 0$$

Now,

$$X'(x) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}x) + \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$$

$$\Rightarrow X'(0) = \sqrt{\lambda}B = 0$$

$$\Rightarrow \lambda = 0 \text{ or } B = 0$$

Case 1:

$B = 0$ and $\lambda \neq 0$ then

$$X'(1) = -\sqrt{\lambda}A\sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \lambda = n^2\pi^2$$

Case 2:

$\lambda = 0$ then $X = A$.

Consequently, we obtain the family of solutions

$$u_0(x,t) = 1$$

$$u_n(x,t) = e^{-n^2\pi^2 t}\cos(n\pi x), \quad n \in \mathbb{N}$$

By linear superposition:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x)$$

Applying Initial Conditions

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$\Rightarrow \langle f(x), 1\rangle = \langle a_0, 1\rangle = \int_0^1 a_0\,dx = a_0.$$

Therefore,

$$\boxed{a_0 = \int_0^1 f(x)\,dx = \text{Area of triangle} = 1/4.}$$

We also have that

$$\langle f(x), \cos(n\pi x)\rangle = \langle a_n\cos(m\pi x), \cos(n\pi x)\rangle = a_n\int_0^1 \cos^2(n\pi x)\,dx = \tfrac{1}{2}a_n.$$

Therefore,

$$a_n = 2\int_0^1 \cos(n\pi x)\,dx$$

$$= 2\left(\int_0^{1/2} x\cos(n\pi x)\,dx + \int_{1/2}^1 (1-x)\cos(n\pi x)\,dx\right)$$

$$= 2\left(\cancel{\frac{x}{n\pi}\sin(n\pi x)\Big|_0^{1/2}} - \int_0^{1/2}\frac{1}{n\pi}\sin(n\pi x)\,dx + \cancel{\frac{1}{n\pi}\sin(n\pi x)\Big|_{1/2}^1}\right.$$

$$\left. \cancel{-\frac{x}{n\pi}\sin(n\pi x)\Big|_{1/2}^1} + \int_{1/2}^1 \frac{1}{n\pi}\sin(n\pi x)\,dx\right)$$

$$= \frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_0^{1/2} - \frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_{1/2}^1$$

$$= \frac{2}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) - \frac{2}{n^2\pi^2}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\boxed{a_n = \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\left(1+(-1)^n\right)}$$

Consequently,

$$u(x,t) = \frac{1}{4} \underbrace{- \frac{2}{\pi^2}e^{-4\pi^2 t}\cos(2\pi x) + \frac{1}{8\pi^2}}_{\text{Dominant Contribution.}} e^{-16\pi^2 t}\cos(4\pi x) + \ldots$$

Steady State Solutions:

Example:

$u_t = u_{xx}$

$u(0,t)=T_1,\ u(1,t)=T_2$ ← rod ends held at constant energy

$u(x,0)=0$ ← No initial heat

*Cannot use seperation of variables because linear superposition fails *

Idea:

Find steady state solution, i.e, solution $u^*(x)$ satisfying

$$u^*(x) = \lim_{t\to\infty} u(x,t)$$

which is a time independent solution:

$$\frac{d^2u^*}{dx^2}=0,\quad u^*(0)=T_1,\ u^*(1)=T_2$$

$$\Rightarrow u^*(x)=(T_2-T_1)x+T_1$$

Now let

$$v(x)=\underbrace{u(x,t)-u^*(x)}$$

v measures seperation from steady state solution.

$$\Rightarrow v_t = u_t = u_{xx}$$

$$v_{xx}=u_{xx}-\cancelto{0}{u^*_{xx}}=u_{xx}$$

Therefore, v satisfies the PDE

$$v_t = v_{xx}$$

$$v(0,t)=0,\ v(1,t)=0$$

$$v(x,0)=-(T_2-T_1)x-T_1$$

Therefore,

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x),$$

where

$$b_n = 2\int_0^1 (-(T_2 - T_1)x - T_1) \sin(n\pi x)\,dx.$$

Therefore,

$$\boxed{u(x,t) = u^*(x) + v(x,t) = (T_2 - T_1)x + T_1 + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x)}$$