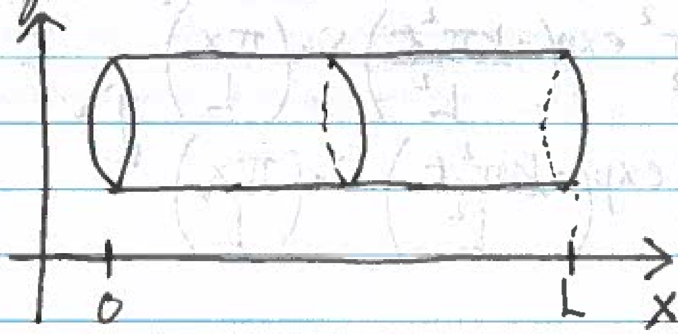
Lecture #2: PDE Models

Heat Flow in a Wire

Let $u(x,t)$ be the temperature in a wire of length L whose ends are maintained at $0°$.



One very useful model says

$$u_t = k \cdot u_{xx}$$

$*\ u_t = \frac{\partial u}{\partial t},\ u_{xx} = \frac{\partial^2 u}{\partial x^2}$

rate of change of temperature, at time t and location x, with respect to time.

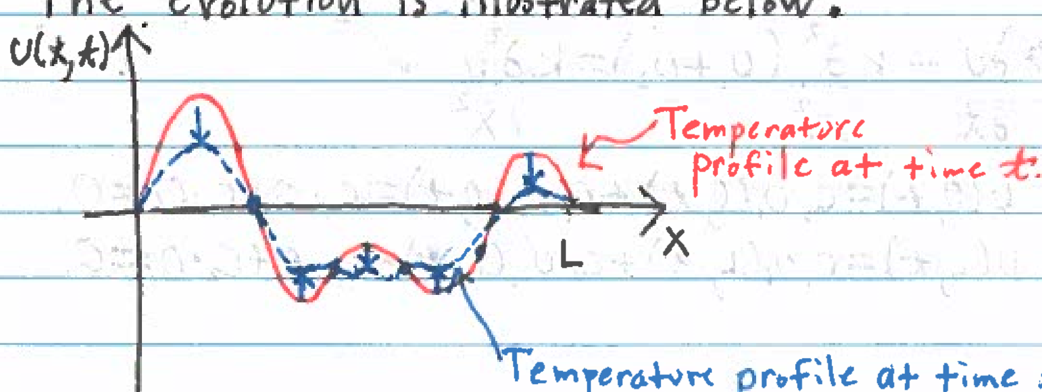
empirical proportionality constant

Concavity/curvature of temperature at time t and location x.

The conditions at $x=0$ and $x=L$ imply the boundary conditions:

$$u(0,t)=0 \text{ and } u(L,t)=0.$$

The evolution is illustrated below:



Can we find solutions?

- One solution is

$$u_1(x,t) = \exp\left(-\frac{k\pi^2 t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right)$$

Check:

$$\frac{\partial u_1}{\partial t} = -\frac{k\pi^2}{L^2}\exp\left(-\frac{k\pi^2 t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right)$$

$$\frac{\partial^2 u_1}{\partial x^2} = -\frac{\pi^2}{L^2}\exp\left(-\frac{k\pi^2 t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right)$$

$$u_1(0,t) = 0$$

$$u_1(L,t) = 0$$

- Another solution is

$$u_2(x,t) = \exp\left(-\frac{4k\pi^2 t}{L^2}\right)\sin\left(\frac{2\pi x}{L}\right)$$

- A whole family of solutions is given by linear superposition:

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t), \quad c_1, c_2 \in \mathbb{R}.$$

Check:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) = c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} = c_1 k\frac{\partial^2 u_1}{\partial x^2} + c_2 k\frac{\partial^2 u_2}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = k\frac{\partial^2}{\partial x^2}(u_1 + u_2) = k\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = c_1 u_1(0,t) + c_2 u_2(0,t) = c_1\cdot 0 + c_2\cdot 0 = 0.$$

$$u(L,t) = c_1 u_1(L,t) + c_2 u_2(L,t) = c_1\cdot 0 + c_2\cdot 0 = 0.$$

Linearity:

Linear superposition worked in this case because the PDE is linear. More precisely, let

$$\begin{matrix} L[u] & =: & \mathcal{L}u \\ \| & & \| \\ u_t - ku_{xx} & = & \left(\frac{\partial}{\partial t} - k\frac{\partial^2 u}{\partial x^2}\right)u. \end{matrix}$$

We say is L is a linear operator if

$$L[u+cv] = L[u] + cL[v] = \mathcal{L}u + c\mathcal{L}v = \mathcal{L}(u+cv).$$

Theorem (Principle of Linear Superposition) - If L is a linear operator and u, v satisfy $L[u] = L[v] = 0$ then for all $c_1, c_2 \in \mathbb{R}$

$$L[c_1 u + c_2 v] = 0.$$

proof

$$L[c_1 u + c_2 v] = c_1 L[u] + c_2 L[v] = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Homogenity:

- A PDE is linear and homogeneous if it can be expressed in the form

$$L[u] = 0$$

- A PDE is linear but inhomogeneous if it can be expressed in the form

$$L[u] = f(t, \underbrace{x_1, \dots, x_n}_{\text{spatial variables}})$$

time variable.

Examples:

1. $u_t + u \cdot u_{xx} = 0$ (nonlinear)
2. $u_{tt} - u_x + \sin(u) = 0$ (nonlinear)
3. $u_t - \sin(x^2 t) u_{xt} = 0$ (linear, nonconstant coefficients)
$$\mathcal{L} u = \left(\frac{\partial}{\partial t} - \sin(x^2 t) \frac{\partial^2}{\partial x \partial t} \right) u$$
4. $u_t + 3x u_{xx} = t x^2$ (linear, inhomogeneous, nonconstant coefficients)
$$\mathcal{L} u = \left(\frac{\partial}{\partial t} + 3x \frac{\partial^2}{\partial x^2} \right) u = t x^2$$

Examples:

1. $u_x = t \sin(x)$
$$\Rightarrow u(x,t) = -t\cos(x) + \psi(t)$$

2. $u_{tt} - 4u = 0$
$$\downarrow$$
like an ode in t

Make a guess of the form:
$$u(x,t) = c(x) e^{\lambda t}$$
$$\Rightarrow \lambda^2 c(x) e^{\lambda t} - 4 c(x) e^{\lambda t} = 0$$
$$\Rightarrow \lambda^2 - 4 = 0$$
$$\Rightarrow \lambda = \pm 2$$

Therefore,
$$u(x,t) = c_1(x) e^{2t} + c_2(x) e^{-2t}$$