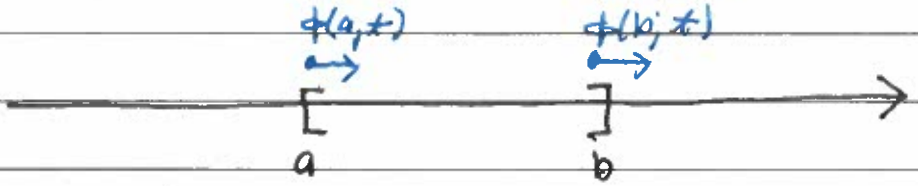
Lecture #3: Conservation Laws

Motivation:

- $x \in \mathbb{R}$, denotes opinion on left/right political spectrum.
- $\rho(x,t)$, denotes density of opinion.
- $\int_a^b \rho(x,t)\,dx$ = Total number of people with opinion in interval (a,b).

- How does opinion change?

$$\boxed{\underbrace{\frac{d}{dt}\int_a^b \rho(x,t)\,dx}_{\text{rate of change of opinion.}} = \phi(a,t) \underset{\substack{\uparrow \\ \text{flux = entering opinion} \\ \text{- exiting opinion}}}{-} \phi(b,t) + \int_a^b \underset{\substack{\uparrow \\ \text{sources of} \\ \text{new voters}}}{f(x,t)}\,dx}$$



ϕ, f determine how ρ changes.

- ϕ could model role of social media
- ϕ could model response to crisis.
- f could model people turning 18
- f could model immigration.

We can simplify using the Fundamental Theorem of Calculus:

$$\phi(a,t) - \phi(b,t) = -\int_a^b \phi_x(x,t)\,dx$$

$$\Rightarrow \int_a^b \rho_t(x,t) + \int_a^b \phi_x(x,t)\,dx = \int_a^b f(x,t)$$

Since, a, b are arbitrary we obtain:

$\boxed{\rho_t + \phi_x = f(x,t)}$ → Conservation law in differential form

Solve for ρ

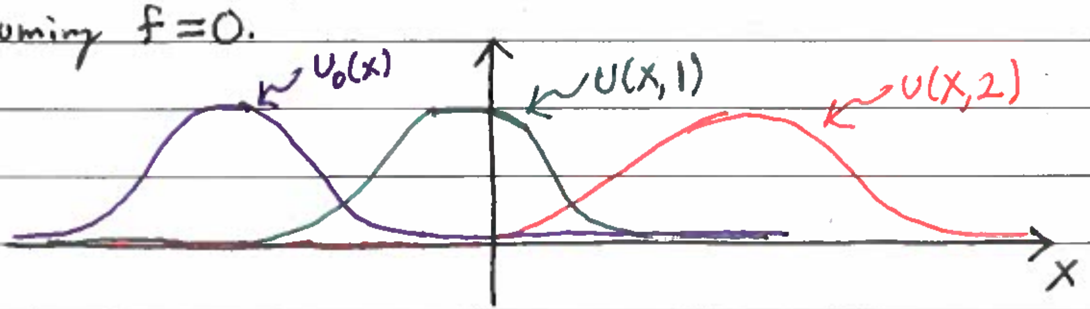
determined by problem we are modeling.

Example:

If $\phi = c\rho$, $c > 0$, we obtain the advection equation

$$\rho_t + c\rho_x = 0,$$
$$u(x,0) = u_0(x),$$

assuming $f = 0$.



Intutively, we expect the density to move to the right at speed c

Make the guess:

$$\rho(x,t) = u_0(x-ct)$$

- $\rho_t = -c\,u_0'(x-ct) \Rightarrow \rho_t + c\rho_x = 0$ ✓
- $\rho_x = u_0'(x-ct)$
- $\rho(x,0) = u_0(x)$ ✓

The guess works, but how do we know it is the only solution?

If we make the change of variables

$$z = x - ct, \quad \tau = t$$

we have that

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -c\frac{\partial}{\partial z} + \frac{\partial}{\partial \tau}$$

With this change of variables we have

$$\rho_t + c\rho_x = (-c\rho_z + \rho_\tau) + c\rho_z = 0$$

$$\Rightarrow \rho_\tau = 0$$

$$\Rightarrow \rho(z,\tau) = f(z)$$

Therefore,

$$\rho(x,t) = f(x-ct).$$

Applying the initial condition

$$\rho(x,0) = f(x) = u_0(x)$$

The unique solution is therefore

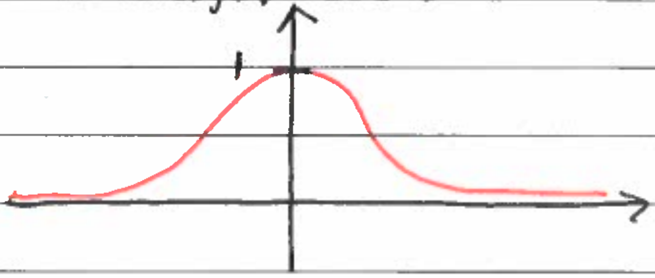
$$\boxed{u(x,t) = u_0(x-ct)}$$

Example:

Sketch a contour plot of solutions to the equation

$$u_t + cu_x = 0 \qquad \Rightarrow u(x,t) = u_0(x-ct).$$

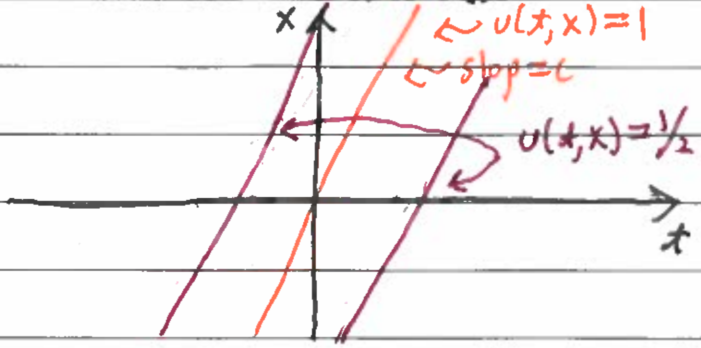
$$u(x,0) = u_0(x)$$



We know that u is constant along the curves

$$x - ct = \text{constant}$$

$$\Rightarrow x = \text{constant} + ct$$



The lines $x = ct + b$ are called characteristic curves. On the characteristic curves, u is constant.

Example:

Solve,

$$u_t + cu_x + au = 0$$

$$u(0,x) = f(x)$$

Let $z = x - ct$, $\tau = t$. Therefore,

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau} - c\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x} = \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial z},$$

$$\Rightarrow u_\tau + au = 0$$

$$\Rightarrow u_\tau = -au$$

$$\Rightarrow u(\tau, z) = g(z)e^{-a\tau}$$

$$\Rightarrow u(t,x) = g(x-ct)e^{-at}.$$

Initial conditions imply

$$u(t,x) = f(x-ct)\, e^{-at}.$$