Lecture #4: Method of Characteristics

Nonuniform Transport

$u_t + c(x)u_x = 0, \quad u(x,0) = f(x)$

wave speed depends on position.

Idea: Track the value of $u(x,t)$ along a curve $(x(t), t)$ in the t-x plane. Define

$$h(t) = u(x(t), t)$$

$$\Rightarrow \frac{dh}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

If $\frac{dx}{dt} = c(x)$ then

$$\frac{dh}{dt} = u_t + c(x)u_x = 0$$

$\Rightarrow u$ is constant along the curve.

The curves $x(t)$ are called characteristic curves.

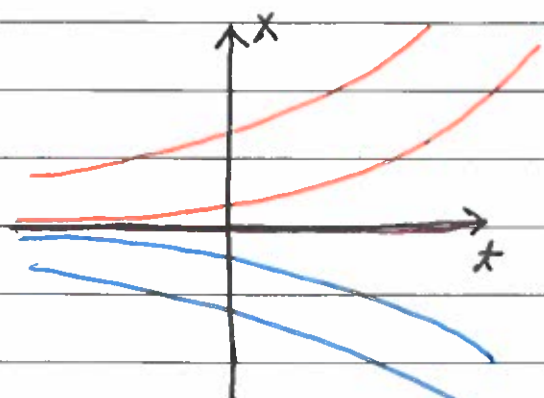
Example:

$u_t + xu_x = 0, \quad u(x,0) = f(x)$

The characteristic curves satisfy

$$\frac{dx}{dt} = x$$

$\Rightarrow x(t) = Ke^t$, where K is constant.



$\Rightarrow$ Solutions spread out in time.

Suppose we want to find a general solution. We know $u(x,t)$ is constant along curves of the form

$$x(t) = Ke^{t}$$

$$\Rightarrow \ln(|x|) - t = \ln(K)$$

The generic solution is a function of $z = \ln(|x|) - t$:

$$\Rightarrow u(t,x) = g(\ln(|x|) - t)$$

Applying boundary conditions

$$u(0,x) = g(\ln(|x|)) = f(x)$$

$$\Rightarrow g(x) = f(e^{x})$$

Therefore,

$$u(t,x) = f(e^{\ln(|x|) - t})$$

$$= f(xe^{-t}).$$

Check:

$$u_t = -xf'(xe^{-t})$$

$$u_x = e^{-t}f'(xe^{-t})$$

$$\Rightarrow u_t + xu_x = 0$$

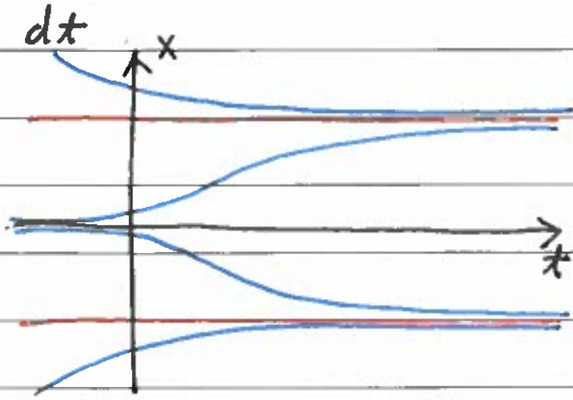
Example:

$$u_t - x(x-1)(x+1)u_x = 0$$

$$u(0,x) = e^{-x^2}$$

Characteristic curves satisfy

$$\frac{dx}{dt} = -x(x-1)(x+1)$$



We can use this information to compute

$$\lim_{t\to\infty} u(x,t) = \begin{cases} 1, & x \in (-1,1) \\ e^{-1}, & x = 0 \\ 0, & x \in (-\infty,-1) \cup (1,\infty) \end{cases}$$