Lecture #5: Dispersion Relationships and Complex Numbers

Example:

Find all solutions to the equation

$$\frac{d^3y}{dx^3} - y = 0, \quad y(0) = a, \quad y'(0) = b, \quad y''(0) = c$$

Approach #1

$$\frac{d^3y}{dx^3} = y$$

Let $\mathcal{L} = \frac{d^3}{dx^3}$

$$\Rightarrow \mathcal{L}y = y$$

This is an eigenvalue problem but $\mathcal{L}$ is not a matrix...

Approach #2

Since the equation is linear and with constant coefficients we make an ansatz of the form

$$y = e^{\lambda x}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \lambda^3 e^{\lambda x}$$

$$\Rightarrow \lambda^3 e^{\lambda x} - e^{\lambda x} = 0$$

$$\Rightarrow \lambda^3 - 1 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, \quad \lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\Rightarrow \lambda = 1, \quad \lambda = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

By linear superposition

$$\boxed{y = c_1 e^{t} + c_2 e^{-\frac{1}{2}t} e^{\sqrt{3}/2\, i t} + c_3 e^{-\frac{1}{2}t} e^{-\sqrt{3}/2\, i t}}$$

: Plug in initial conditions:

$$a = c_1 + c_2 + c_3$$
$$b = c_1 + \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)c_2 + \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)c_3$$
$$c = c_1 + \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right)^2 c_2 + \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)^2 c_3$$

How can we make sense of this. Existence and uniqueness says for all a, b, c we should have unique real valued solutions?? We need to learn some complex numbers...

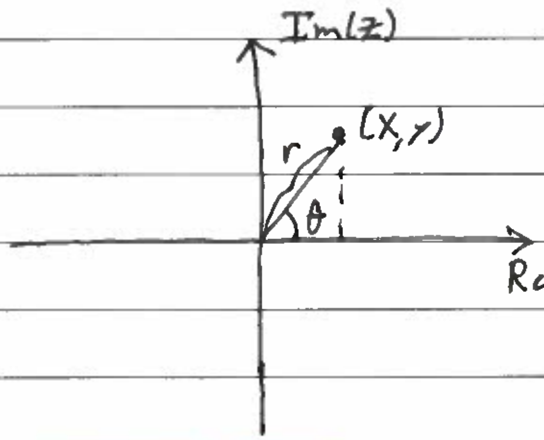
Complex Numbers

Complex number is any number of the form

$\boxed{z = x + iy}$ (Cartesian Form)

$x, y \in \mathbb{R}$.

We can visualize complex numbers in the complex plane as coordinates



$$\Rightarrow \boxed{\begin{array}{l} z = r(\cos\theta + i\sin\theta) \\ r = |z| = \sqrt{x^2 + y^2} \end{array}}$$ (Polar Form)

Moreover,

$$\cos\theta = 1 - \theta^2/2 + \theta^4/4! + \ldots$$
$$\sin\theta = \theta - \theta^3/3! + \theta^5/5! + \ldots$$
$$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$$
$$\Rightarrow e^{i\theta} = 1 + i\theta + (i\theta)^2/2! + (i\theta)^3/3! + \ldots$$

Therefore,

$$e^{i\theta} = 1 + i\theta - \theta^2/2! - i\theta^3/3! + \theta^4/4! + \dots$$
$$= (1 - \theta^2/2! + \theta^4/4! + \dots) + i(\theta - \theta^3/3! + \theta^5/5! + \dots)$$
$$= \cos\theta + i\sin\theta$$

Consequently, we can express complex numbers in an exponential form:

$$\boxed{z = re^{i\theta} = |z|e^{i\theta}} \quad \text{(Exponential Form)}$$

Return to ODE Example

$$y = c_1 e^t + c_2 e^{-1/2 t}(\cos(\sqrt{3}/2\, t) + i\sin(\sqrt{3}/2\, t)) + c_3 e^{-1/2 t}(\cos(\sqrt{3}/2\, t) - i\sin(\sqrt{3}/2\, t))$$

$$\Rightarrow y = c_1 e^t + (c_2 + c_3)e^{-1/2 t}\cos(\sqrt{3}/2\, t) + i(c_2 - c_3)\sin(\sqrt{3}/2\, t)$$

In order to be a real solution, we need

$$c_2 + c_3 = Re(c_2) + Re(c_3) + i(Im(c_2) + Im(c_3)) \in \mathbb{R}$$
$$i(c_2 - c_3) = i(Re(c_2) - Re(c_3)) - (Im(c_2) - Im(c_3)) \in \mathbb{R}$$

Therefore,

$$Im(c_2) + Im(c_3) = 0$$
$$Re(c_2) - Re(c_3) = 0$$
$$\Rightarrow Re(c_2) = Re(c_3) \text{ and } Im(c_2) = -Im(c_3)$$

Consequently,

$$c_3 = \bar{c}_2,$$

where the complex conjugate of a complex number is defined by

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

The main takeaway is we can write the generic solution in the form:

$$\boxed{y = d_1 e^t + d_2 e^{-1/2 t}\cos(\sqrt{3}/2\, t) + d_3 e^{-1/2 t}\sin(\sqrt{3}/2\, t)},$$

where $d_1, d_2, d_3 \in \mathbb{R}$.

Dispersion Relationships

Return to the heat equation

$$u_t = D u_{xx}, \quad u(x,0) = f(x).$$

How can we guess a solution? This is a linear PDE with constant coefficients. So, we make a guess of the form

$$u = e^{w \cdot t + ikx}$$
$$= e^{wt}(\cos(kx) + i\sin(kx))$$

We obtain

$$w e^{wt} e^{ikx} = (ik)^2 e^{wt} e^{ikx}$$
$$\Rightarrow w = -k^2 \quad \text{(Dispersion Relationship)}$$

Therefore, a solution is given by

$$u_k(x,t) = e^{-k^2 t} e^{ikx}, \quad k \in \mathbb{R}.$$

For an arbitrary function $g(k)$ we have by linear superposition

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} g(k) e^{-k^2 t} e^{ikx} dk,}$$

is also a solution. From initial conditions we have

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk$$

The theory of Fourier transforms is designed to solve this problem.