Lecture #6: Fourier Transforms

The Fourier transform of a function $f(x)$, $x \in \mathbb{R}$ is defined by:

$$\underset{\substack{\uparrow \\ \text{linear} \\ \text{transformation}}}{\mathcal{F}}[u](k) = \underset{\substack{\uparrow \\ \text{Fourier transform} \\ \text{notation}}}{\hat{u}(k)} = \underbrace{\int_{-\infty}^{\infty} u(x) e^{ikx} dx}_{\uparrow \text{definition.}}$$

Since $e^{ikx} = \cos(kx) + i\sin(kx)$, the Fourier transform is a linear superposition of periodic functions with period

$$\lambda = \frac{2\pi}{k},$$

k is called the wavenumber. The inverse Fourier transform of a function $v(k)$, $k \in \mathbb{R}$ is defined by:

$$\underset{\substack{\uparrow \\ \text{linear} \\ \text{transformation}}}{\mathcal{F}^{-1}}[v](x) = \underset{\substack{\uparrow \\ \text{inverse} \\ \text{Fourier transform} \\ \text{notation}}}{\check{v}(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{spectrum}}}{v(k)} e^{-ikx} dk$$

Proposition: If $\int_{-\infty}^{\infty} |u(x)| dx < \infty$ then $\hat{u}(k)$ exists.

proof:

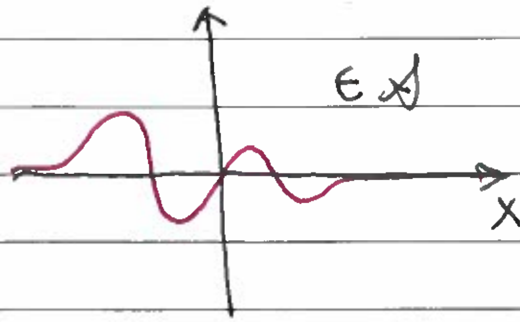
$$|\hat{u}(k)| = \left|\int_{-\infty}^{\infty} u(x) e^{ikx} dx\right| \le \int_{-\infty}^{\infty} |u(x) e^{ikx}| dx = \int_{-\infty}^{\infty} |u(x)| dx < \infty$$

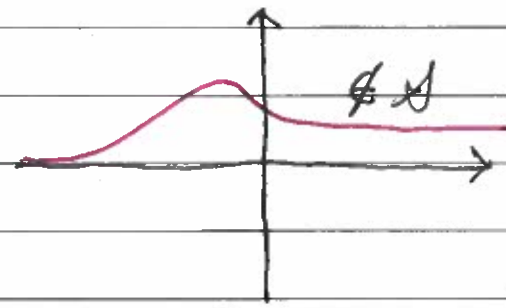
$\blacksquare$

Assumptions:

1. u is infinitely differentiable.
2. $\lim_{x \to \infty} \frac{|u^{(k)}(x)|}{|x|^N} = 0$, for all $k, N \in \mathbb{N}$.

The set $\mathcal{S}$ of all functions satisfying these properties is called Schwartz class.





Properties:

If $u \in \mathcal{S}$ then

1. $\mathcal{F}^{-1}[\mathcal{F}[u]] = u$
2. $\mathcal{F}[\mathcal{F}^{-1}[u]] = u$
3. $\mathcal{F}[u+\lambda v] = \mathcal{F}[u] + \lambda \mathcal{F}[v]$
4. $\mathcal{F}[u_x] = \int_{-\infty}^{\infty} u_x e^{ikx}dx = \cancelto{0}{ue^{ikx}\Big|_{-\infty}^{\infty}} - ik\int_{-\infty}^{\infty} ue^{ikx}dx$

   $\Rightarrow \mathcal{F}[u_x] = -ik\hat{u}$
5. $\mathcal{F}[u_{xx}] = -k^2\hat{u}$
6. $\mathcal{F}[u_t] = \int_{-\infty}^{\infty} u_t e^{ikx}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{ikx}$

   $\Rightarrow \mathcal{F}[u_t] = \hat{u}_t.$

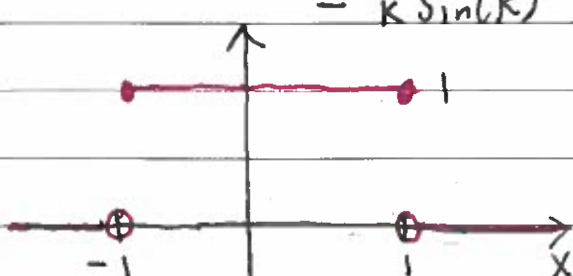
Examples:

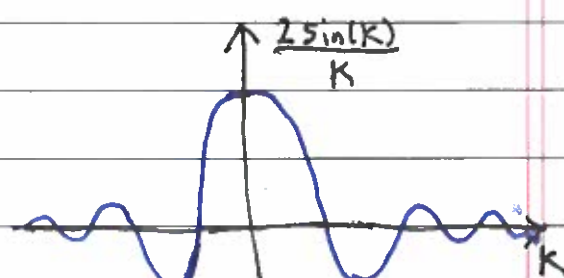
1. Let

$$u(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & x > 1 \text{ or } x < -1 \end{cases}$$

Therefore,

$$\begin{aligned} \mathcal{F}[u] &= \int_{-\infty}^{\infty} u(x)e^{ikx}dx \\ &= \int_{-1}^{1} (\cos(kx) + i\sin(kx))dx \\ &= 2\int_{0}^{1} \cos(kx)dx \\ &= \frac{2}{k}\sin(k) \end{aligned}$$





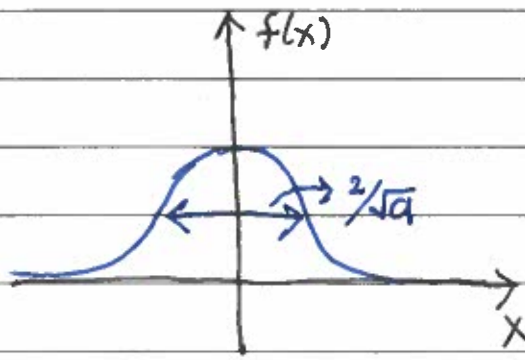
2. Let $f(x)=e^{-ax^2}$.

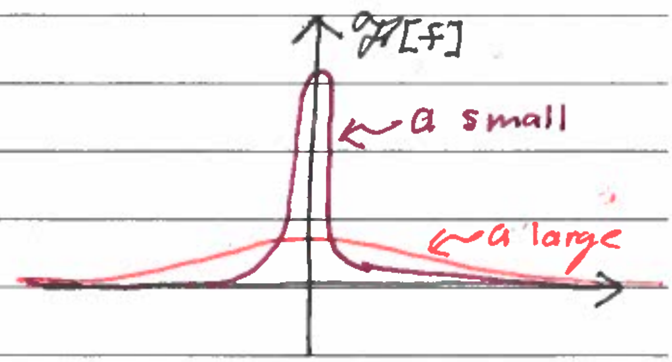
$$\mathcal{F}[f]=\hat{f}(k)=\int_{-\infty}^{\infty}e^{-ax^2}e^{ikx}\,dx$$
$$=\int_{-\infty}^{\infty}e^{-a(x^2-ikx/a)}\,dx$$
$$=\int_{-\infty}^{\infty}e^{-a(x^2-ikx/a-k^2/4a^2+k^2/4a^2)}\,dx$$
$$=\int_{-\infty}^{\infty}e^{-a(x-ik/2a)^2}e^{-k^2/4a}\,dx$$
$$=e^{-k^2/4a}\int_{-\infty}^{\infty}e^{-a(x-ik/2a)^2}\,dx$$

Letting $u=\sqrt{a}\,(x-ik/2a)$

$$\mathcal{F}[f]=\frac{e^{-k^2/4a}}{\sqrt{a}}\int_{-\infty}^{\infty}e^{-u^2}\,du$$

$$\boxed{\mathcal{F}[e^{-ax^2}]=\sqrt{\frac{\pi}{a}}\,e^{-k^2/4a}}$$





Convolution Theorem

If $u, v \in \mathcal{S}$, the convolution is defined by

$$(u*v)(x)=\int_{-\infty}^{\infty}u(x-y)v(y)\,dy$$

Example:

Let $f(x)=\begin{cases}1, & |x|\le 1\\ 0 & |x|>1\end{cases}$

$g(x)=e^{-|x|}$.

Therefore,

$$f*g = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$
$$= \int_{-1}^{1} e^{-|x-y|}dy$$

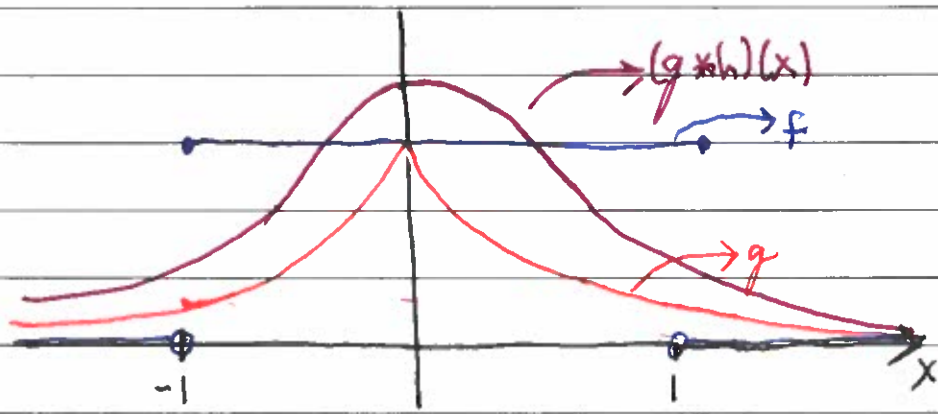
Let $u = x-y$,

$$f*g = -\int_{x+1}^{x-1} e^{-|u|}du$$

$$= \int_{x-1}^{x+1} e^{-|u|}du$$

$$\Rightarrow f*g = \begin{cases} \int_{x-1}^{x+1} e^{u}du, & x+1<0 \\ \int_{x-1}^{0} e^{u}du + \int_{0}^{x+1} e^{-u}du, & x+1>0,\ x-1<0 \\ \int_{x-1}^{x+1} e^{-u}du, & x-1>0 \end{cases}$$

$$\Rightarrow f*g = \begin{cases} e^{x+1}-e^{x-1}, & x<-1 \\ 2-e^{x-1}-e^{-x-1}, & -1<x<1 \\ e^{1-x}-e^{-x-1}, & 1<x \end{cases}$$



Theorem - If $u, v \in \mathcal{S}$ then

1. $\mathcal{F}[u*v] = \mathcal{F}[u]\cdot\mathcal{F}[v] = \hat{u}(k)\hat{v}(k)$
2. $\mathcal{F}^{-1}[\hat{u}\hat{v}] = u*v$

proof

$$\mathcal{F}[u*v] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x-y)v(y)e^{ikx}dydx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x-y)v(y)e^{ikx}dxdy$$

Let $w = x - y$,

$$\mathcal{F}[u * v] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(w)v(y)e^{ikw}e^{iky}\,dw\,dy$$

$$= \mathcal{F}[u]\cdot\mathcal{F}[v].$$