Lecture #7: Solving PDEs using Fourier Transforms

Example:

$$u_t + c u_x = 0$$
$$u(x,0) = u_0(x)$$

Take Fourier transform of both sides

$$\hat{u}_t = ikc\hat{u}$$
$$\hat{u}(k,0) = \hat{u}_0(k)$$
$$\Rightarrow \hat{u}(k,t) = \hat{u}_0(k) e^{ikct}$$

$$\Rightarrow u(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_0(k) e^{ikct} e^{-ikx} dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_0(k) e^{-ik(x-ct)} dk$$
$$= u_0(x-ct).$$

Example:

$$u_t = D u_{xx}$$
$$u(x,0) = f(x)$$

Take Fourier transform of both sides

$$\hat{u}_t = -k^2 \hat{u}$$
$$\hat{u}(k,0) = \hat{f}(k)$$

$$\Rightarrow \hat{u}(k,t) = \hat{f}(k) e^{-k^2 t}$$
$$\Rightarrow u(x,t) = f(x) * \mathcal{F}^{-1}[e^{-k^2 t}](x)$$

Now,

$$\mathcal{F}^{-1}[e^{-k^2t}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2t}e^{-ikx}dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^2t}e^{iux}du$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\, e^{-x^2/4t}$$

$$= \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$$

Consequently, a solution is given by:

$$u(x,t) = f(x) * e^{-x^2/4t}/\sqrt{4\pi t}$$

$$\Rightarrow \boxed{u(x,t) = \int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{4\pi t}}e^{-(x-y)^2/4t}dy}$$

The function

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$$

is called

1. The heat kernel
2. Green's function
3. Fundamental solution.

Properties of Heat Equation:

$$u_t = Du_{xx} \quad , \quad u \in \mathcal{A}.$$
$$u(x,0) = f(x)$$

1. $\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

proof:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t\,dx = \int_{-\infty}^{\infty} u_{xx}\,dx = u_x(\infty) - u_x(-\infty) = 0.$$

Therefore, $\int_{-\infty}^{\infty} u(x,t)\,dx$ is constant and thus

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

2. Let $E(t) = \int_{-\infty}^{\infty} u(x,t)^2\,dx$. Then $\frac{d}{dt}E(t) \leq 0$.

proof:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} 2uu_t\,dx = \int_{-\infty}^{\infty} 2u \cdot u_{xx}\,dx = 2u\,\cancel{u_{xx}}\,u_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2u_x^2\,dx$$

$$\Rightarrow \frac{dE}{dt} = -2\int_{-\infty}^{\infty} u_x^2\,dx \leq 0.$$

3. Solutions are unique

proof:

Let u, v be solutions. By linearity, if we let $w = u - v$ then w satisfies

$$w_t = Dw_{xx}$$
$$w(x,0) = 0$$

If we let $E = \int_{-\infty}^{\infty} w^2\,dx$, then

1. $E(0) = 0$
2. $E(t) \geq 0$
3. $\frac{dE}{dt} \leq 0$

Consequently, $E \equiv 0$ which implies

$$0 = \int_{-\infty}^{\infty} w^2\,dx \Rightarrow w = 0.$$

4. If $|f(x)|\le M$ then $|u(x,t)|\le M$.

~~proof~~:

$$|u(x,t)| = \left|\int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/4\pi Dt} \cdot \frac{1}{\sqrt{4\pi Dt}}\,dy\right|$$

$$\le \int_{-\infty}^{\infty} |f(y)|\, e^{-(x-y)^2/4\pi Dt} \cdot 1/\sqrt{4\pi Dt}\,dy$$

$$\le M\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}}\, e^{-(x-y)^2/4\pi Dt}\,dy$$

$$= M.$$

Example:

Solve

$$U_t + cU_x = DU_{xx}, \quad u(x,0) = f(x)$$

where $c\in\mathbb{R}$, $D>0$. Take Fourier transform of both sides

$$\hat{U}_t - ick\hat{U} = -k^2D\hat{U}$$

$$\hat{U}(k,0) = \hat{f}(k)$$

$$\Rightarrow \hat{U}_t = (ick - k^2D)\hat{U}$$

$$\Rightarrow \hat{U} = \hat{f}(k)\, e^{(ick-k^2D)t}$$

Therfore,

$$U = \mathcal{F}^{-1}[\hat{f}(k)e^{ickt}] * \mathcal{F}^{-1}[e^{-k^2Dt}]$$

$$= f(x-ckt) * e^{-k^2Dt}$$

Thus

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} f(y-ckt)\, e^{-(x-y)^2/4Dt}\,dy.$$