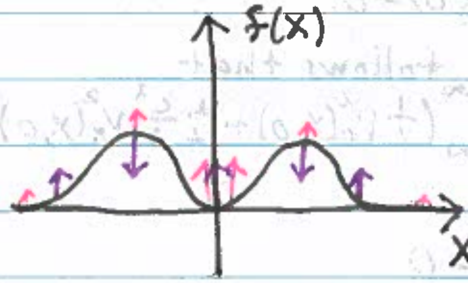
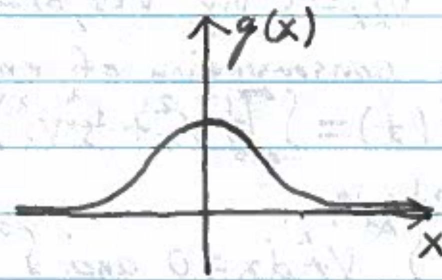
Lecture #8: Wave Equation on the Line

$$U_{tt} = c^2 U_{xx}$$
$$U(x,0) = f(x) \qquad (*)$$
$$U_t(x,0) = g(x)$$





*velocity
*accleration

Properties:

1. Conservation of Energy:

Define
$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} U_t^2\,dx + \frac{c^2}{2}\int_{-\infty}^{\infty} U_x^2\,dx$$

Then
$$\frac{dE}{dt} = 0.$$

proof:
$$\frac{dE}{dt} = \int_{-\infty}^{\infty} U_t U_{tt}\,dx + c^2\int_{-\infty}^{\infty} U_x U_{xt}\,dx$$
$$= \int_{-\infty}^{\infty} c^2 U_t U_{xx}\,dx + c^2\int_{-\infty}^{\infty} U_x U_{xt}\,dx$$
$$= c^2 \overset{0}{\cancel{U_t U_x\Big|_{-\infty}^{\infty}}} - c^2\int_{-\infty}^{\infty} U_{tx} U_x\,dx + \int_{-\infty}^{\infty} U_x U_{xt}\,dx$$
$$= 0. \qquad \blacksquare$$

2. Theorem - Solutions to $(*)$ are unique.

~~proof~~:

Suppose u_1, u_2 solve $(*)$ and let $v = u_2 - u_1$. By linearity, it follows that v satisfies

$$v_{tt} = c^2 v_{xx}, \; v(x,0) = 0, \; v_t(x,0) = 0.$$

From conservation of energy it follows that

$$E(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2} v_t^2 + \frac{c^2}{2} v_x^2\right) dx = \int_{-\infty}^{\infty} \left(\frac{1}{2} v_t^2(x,0) + \frac{1}{2}\frac{c^2}{2} v_x^2(x,0)\right) dx = 0.$$

Therefore,

$$\int_{-\infty}^{\infty} v_t^2 \, dx = 0 \text{ and } \int_{-\infty}^{\infty} v_x^2 \, dx = 0$$

$$\Rightarrow v_t = 0 \text{ and } v_x = 0$$

$$\Rightarrow v = \text{constant}$$

Since $v(x,0) = 0$ it follows that $v(x,t) = 0$

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Solution:

Consider

$$u_{tt} = c^2 u_{xx}.$$

Let

$$\xi = x - ct, \quad \eta = x + ct$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} = \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)$$

$$= c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\frac{\partial^2}{\partial \xi \partial \eta} + c^2\frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}$$

Therefore,

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$
$$\Rightarrow 4u_{\xi\eta} = 0$$
$$\Rightarrow u_{\xi\eta} = 0$$
$$\Rightarrow u_\xi = \phi(\xi)$$
$$\Rightarrow u = \Phi(\xi) + \theta(\eta),$$

where $\Phi(\xi)$, $\theta(\eta)$ are arbitrary functions. Consequently, by uniqueness, all solutions are of the form

$$u(x,t) = \Phi(x-ct) + \theta(x+ct).$$

Fundamental Solution:

$$u_{tt} = c\, u_{xx}$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

Now,

$$u(x,t) = \Phi(x-ct) + \theta(x+ct)$$
$$u_t(x,t) = -c\Phi'(x-ct) + c\theta'(x-ct)$$

Therefore,

$$\Phi(x) + \theta(x) = f(x)$$
$$-c\Phi'(x) + c\theta'(x) = g(x)$$
$$\Rightarrow c\Phi'(x) + c\theta'(x) = cf'(x)$$
$$-c\Phi'(x) + c\theta'(x) = g(x)$$

Consequently,

$$\theta'(x) = \frac{1}{2} f'(x) + \frac{1}{2c} g(x)$$

$$\Phi'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x)$$

$$\Rightarrow \theta(x) = \frac{1}{2}\int_{-\infty}^{x} f'(s)\,ds + \frac{1}{2c}\int_{-\infty}^{x} g(s)\,ds$$
$$= \frac{1}{2}f(x) + \frac{1}{2c}\int_{-\infty}^{x} g(s)\,ds.$$

$$\Phi(x) = \frac{1}{2}\int_{-\infty}^{x} f'(s)\,ds - \frac{1}{2c}\int_{-\infty}^{x} g(s)\,ds$$
$$= \frac{1}{2}f(x) - \frac{1}{2c}\int_{-\infty}^{x} g(s)\,ds$$

Therefore,

$$u(x,t) = \theta(x+ct) + \Phi(x-ct)$$
$$= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{-\infty}^{x+ct} g(s)\,ds + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{-\infty}^{x-ct} g(s)\,ds$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.}$$

This is known as d'Alembert's formula.

Example:

Consider the initial value problem:

$$u_{tt} = c^2 u_{xx} \qquad f(x) = \begin{cases} x+1, & -1<x<0 \\ 1-x, & 0 \le x < 1 \\ 0, & |x| \ge 1 \end{cases}$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

The solution is

$$u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct).$$