Lecture #9: The Fourier Method

Recall:

To solve the initial value problem

$$U_t = U_{xx}, \quad x \in \mathbb{R}, \; t > 0$$
$$U(x,0) = f(x)$$

we used Fourier transforms:

$$\hat{U}_t = -k^2 \hat{U}$$
$$\hat{U}(k,0) = \hat{f}$$
$$\Rightarrow \hat{U} = \hat{f} e^{-k^2 t}$$

Therefore,

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{f}(k)}_{\text{amplitude of basis functions from initial conditions}} \underbrace{e^{-k^2 t} e^{-ikx}}_{\text{basis functions are decaying waves of the form}} dk$$

$$b_k(x,t) = e^{-k^2 t}(\cos(kx) - i\sin(kx))$$

Example:

$$U_t = U_{xx}, \quad 0 < x < \pi$$
$$U(0,t) = U(\pi,t) = 0 \rightarrow \text{Boundary conditions}$$
$$U(x,0) = f(x) \rightarrow \text{Initial heat}$$

Look for linear combination of solutions:

(a) $b_k(x,t) = e^{-at}\sin(kx)$

$$\Rightarrow -a e^{-at}\sin(kx) = -k^2 e^{-at}\sin(kx)$$
$$\Rightarrow a = k^2$$
$$\Rightarrow b_k(x,t) = e^{-k^2 t}\sin(kx)$$

(b) A generic solution can be written in the form

$$u(x,t)=\sum_{k=1}^{\infty}\underbrace{a_k}_{\text{components}}\underbrace{e^{-k^2t}\sin(kx)}_{\text{basis}},$$

boundary conditions imply $k\in\mathbb{N}$.

(c) Applying initial conditions we have that

$$u(x,0)=\sum_{k=1}^{\infty}a_k\sin(kx)=f(x)$$

For $m\in\mathbb{N}$ we have:

$$\int_0^{\pi}\sum_{k=1}^{\infty}a_k\sin(kx)\sin(mx)\,dx=\int_0^{\pi}f(x)\sin(mx)\,dx$$

$$\Rightarrow\sum_{k=1}^{\infty}\int_0^{\pi}a_k\sin(kx)\sin(mx)\,dx=\int_0^{\pi}f(x)\sin(mx)\,dx$$

Now, if $m\neq n$ we have:

$$\int_0^{\pi}\sin(kx)\sin(mx)\,dx=\frac{1}{(2i)^2}\int_0^{\pi}(e^{ikx}-e^{-ikx})(e^{imx}-e^{-imx})\,dx$$

$$=-\frac{1}{4}\int_0^{\pi}\left(e^{i(k+m)x}-e^{i(k-m)x}-e^{-i(k-m)x}+e^{-i(k+m)x}\right)dx$$

$$=\frac{1}{2}\int_0^{\pi}\cos((k-m)x)\,dx-\frac{1}{2}\int_0^{\pi}\cos((k+m)x)\,dx$$

$$=\frac{1}{2(k-m)}\sin((k-m)x)\Big|_0^{\pi}-\frac{1}{2(k+m)}\sin((k+m)x)\Big|_0^{\pi}$$

$$=0$$

If $m=n$ we have:

$$\int_0^{\pi}\sin^2(kx)\,dx=\frac{1}{2}\int_0^{\pi}dx-\frac{1}{2}\int_0^{\pi}\cos(2kx)\,dx$$

$$=\frac{\pi}{2}.$$

Therefore,

$$\frac{\pi}{2} a_m = \int_0^{\pi} f(x)\sin(mx)\,dx$$

dummy variable

$$\Rightarrow a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$$

The general solution is therefore

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-k^2 t}\sin(kx)$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}\int_0^{\pi}\left(f(y)\sin(ky)\right)dy\, e^{-k^2 t}\sin(kx).$$

Comments:

1. Does the infinite series converge?
2. Is this the only solution?
3. How do we find the basis functions?