Theorem 2.20 implies that E^c is the complement of E^s, since the generalized eigenvectors span $\mathbb{R}^3$. To demonstrate this, find generalized eigenvectors by solving

$$(A-0I)^2 v = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 4 & 4 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.$$

This is equivalent to the single equation $a+b+c=0$, so that there are two arbitrary constants in v (we knew this already since $\dim(E^c)=2$). One representation of the solution is $v = av_2 + bv_3$, where $v_2 = (1,0,-1)^T$ and $v_3 = (0,1,-1)^T$. Consequently,

$$E^c = \text{span}(v_2, v_3) = \left\{ \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Finally we ask, is the system linearly stable? For this to be the case, the nilpotent part of A must vanish, or alternatively there must be two independent eigenvectors corresponding to $\lambda = 0$. The eigenvalue problem $(A-0I)v = 0$ has only a single solution, $v = (1,0,-1)^T$. Since the nilpotent part is nonzero our system is not linearly stable. This is confirmed by finding

$$S = P\Lambda P^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & 0 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix},$$

giving a nilpotent part

$$N = A - S = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

which is easily seen to satisfy $N^2 = 0$. Finally, the exponential is

$$e^{tA} = Pe^{t\Lambda}P^{-1}(I+tN) = \frac{1}{2} \begin{pmatrix} e^{-2t}+1-2t & e^{-2t}-1 & e^{-2t}-1-2t \\ 2e^{-2t}-2 & 2e^{-2t} & 2e^{-2t}-2 \\ -e^{-2t}+1+2t & -e^{-2t}+1 & -e^{-2t}+3+2t \end{pmatrix},$$

confirming that this system is unstable since there are terms that grow linearly in time. In particular, if $x_o = (1,0,0)^T$, then $x(t) \to 2t(-1,0,1)^T \to \infty$. Note that not all solutions are unbounded. For example, if $x_o = (0,1,0)^T$, then $x(t) \to (-1,0,1)^T$. Nevertheless, a single unbounded solution is enough to declare the system unstable. ∎

2.8 ▪ Nonautonomous Linear Systems and Floquet Theory

A linear physical system that is externally forced can often be modeled by the affine set of ODEs,

$$\dot{x} = Ax + f(t).$$

Such differential equations can be easily solved using the "integrating factor" method; see Exercise 17. It is considerably more difficult to solve a linear system when the matrix A depends upon time,

$$\dot{x} = A(t)x, \; x(t_o) = x_o. \tag{2.45}$$

Nonautonomous equations like these can arise in mechanical systems if the forcing changes the effective spring constants; for example, a person pumping his legs on a swing will change the effective length of the pendulum and thereby modulate the coefficient g/l that governs the linear oscillation frequency. Equations of the form (2.45) also occur as the linearization of the dynamics about a periodic orbit of period T. In this case the matrix A is a periodic function of time, $A(t+T) = A(t)$. Gaston Floquet developed the theory of the solutions of such systems in the 1880s (Chicone 1999, §2.4; Floquet 1883; Yakubovitch and Starzhinskii 1975, Chapter 5).

To solve (2.45), it is convenient to consider a matrix differential equation of the form (2.33), replacing the vector $x(t)$ by a matrix. The general solution is most conveniently represented in terms of the principal *fundamental matrix* solution, which is the solution $\Phi(t, t_o)$ of the matrix initial value problem

$$\frac{d}{dt}\Phi = A(t)\Phi, \quad \Phi(t_o, t_o) = I. \tag{2.46}$$

Here we have added a second argument to Φ to indicate that the initial condition is applied at time t_o. As for the autonomous case, the solution of the original system with initial value $x(t_o) = x_o$ is simply given by $x(t) = \Phi(t, t_o)x_o$. Thus, if we can find $\Phi(t, t_o)$, we also have the general solution to (2.45). We will ignore for the moment the more delicate question of the existence and uniqueness of Φ; this will follow more generally from Theorem 3.24, requiring only that $A(t)$ be a continuous function of time. Uniqueness implies that the fundamental matrix solution obeys the relation

$$\Phi(t, r) = \Phi(t, s)\Phi(s, r) \tag{2.47}$$

for all $t, s, r \in \mathbb{R}$.

When A is constant $\Phi(t, t_o) = e^{(t-t_o)A}$, and we proved in §2.4 that this is the unique solution. However, this formula no longer works for the time-dependent case, and more importantly, the "obvious" generalization

$$\Phi(t, t_o) = \exp\left(\int_{t_o}^{t} A(s)ds\right) \quad \text{(incorrect!)} \tag{2.48}$$

is usually wrong since the matrix $A(s_1)$ does not generally commute with $A(s_2)$ when $s_1 \neq s_2$ (see Exercises 18–19). Moreover, as the following example shows, the eigenvalues of the matrix $A(t)$ at a fixed value of time may have nothing to do with the properties of the solution of (2.45).

Example 2.32. Here is an example that points out the pitfalls of looking at the eigenvalues of $A(t)$ (Markus and Yamabe 1960). Consider the time-dependent matrix

$$A(t) = \begin{pmatrix} -1 + \alpha\cos^2 t & 1 - \alpha\cos t \sin t \\ -1 - \alpha\cos t \sin t & -1 + \alpha\sin^2 t \end{pmatrix}.$$

It is easy to see that the eigenvalues of this matrix are independent of time because $\mathrm{tr}(A) = \alpha - 2$, and $\det(A) = 2 - \alpha$, so

$$\lambda = \tfrac{1}{2}\left(\alpha - 2 \pm \sqrt{\alpha^2 - 4}\right).$$

When $\alpha < 2$, the eigenvalues indicate that this system may be stable. However, the differential equation $\dot{x} = A(t)x$ has two simple explicit solutions, as can be easily verified by substitution:

$$x_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{(\alpha-1)t}, \quad x_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^{-t}. \tag{2.49}$$

Therefore, when $\alpha > 1$ the first solution is unbounded and thus the system is unstable. Consequently, for the range $1 < \alpha < 2$ the system is unstable, even though the eigenvalues of $A(t)$ would suggest that it should be stable. This example shows that the eigenvalues of a nonautonomous matrix do not generally determine the stability of the corresponding ODE. ∎

For the case that A is a periodic matrix, an important quantity is the value of the fundamental matrix at one period; it is called the

▷ *monodromy matrix*, $M \equiv \Phi(T,0)$.

Given the initial condition $x(0) = x_o$, then $x(T) = Mx_o$. To continue this solution past T requires finding the solution of the initial value problem

$$\dot{x} = A(t)x, \quad x(T) = Mx_o.$$

Define a new time variable $\tau = t - T$, and use $A(\tau + T) = A(\tau)$ to see that this is the same as the initial value problem (2.45), with x_o replaced by Mx_o, so its solution is $\Phi(\tau,0)Mx_o$. This implies

$$x(2T) = M^2 x_o.$$

In consequence, to get the long-time behavior of any solution, we merely need to compute M^n.

The eigenvalues of M are called the *Floquet multipliers*. Suppose x_o is an eigenvector of M with eigenvalue μ; then

$$x(nT) = \mu^n x_o = e^{n \ln \mu} x_o.$$

The exponent $\ln \mu$ is called a *Floquet exponent*; it is a special case of the Lyapunov exponent that we will meet in Chapter 7.

Example 2.33. Continuing the previous example, note that the matrix $A(t)$ is periodic with period $T = \pi$. Moreover, the two solutions (2.49) are linearly independent, and since $x_1(0) = (1,0)^T$ and $x_2(0) = (0,1)^T$, the fundamental solution is $\Phi(t,0) = [x_1(t), x_2(t)]$. Evaluating this at $t = \pi$ gives the monodromy matrix

$$M = \Phi(\pi,0) = \begin{pmatrix} -e^{\pi(\alpha-1)} & 0 \\ 0 & -e^{-\pi} \end{pmatrix},$$

showing that the Floquet multipliers are $\mu_1 = -e^{\pi(\alpha-1)}$ and $\mu_2 = -e^{-\pi}$. Note that when $\alpha > 1$, there is one Floquet multiplier with magnitude larger than one and one with magnitude smaller than one. ∎

In general, the monodromy matrix M is nonsingular. In fact, there is a simple equation for the evolution of the determinant of Φ that holds even when $A(t)$ is not periodic. This theorem generalizes the standard result by Abel for the "Wronskian" of a second-order ODE.

Theorem 2.34 (Abel). *The determinant of the fundamental matrix is*

$$\det(\Phi(t,t_0)) = \exp\int_{t_0}^{t} \operatorname{tr}(A(s))ds. \tag{2.50}$$

Note that $\operatorname{tr}(A(s))$ *is a scalar, so the exponential is the ordinary, scalar exponential.*

Proof. Our goal is to obtain a simple ODE for $\det(\Phi)$. The derivative of the determinant of Φ can be computed using the cofactor formula. Recall that the cofactor, c_{ij}, is $(-1)^{i+j}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained by omitting the ith row and the jth column from Φ. Multiplying c_{ij} by Φ_{ij} and summing over j, i.e., summing along the ith row, gives

$$\det(\Phi) = \sum_{j=1}^{n} c_{ij}\Phi_{ij}.$$

This formula is true for any choice of row i. If instead we multiply c_{ij} by Φ_{kj}, and then sum over j, then this is equivalent to computing the determinant of the matrix with the ith row replaced by the kth row. Since the resulting matrix has two equal rows, its determinant is zero. This generalization of the cofactor formula can be written as

$$\det(\Phi)\ \delta_{ik} = \sum_{j=1}^{n} c_{ij}\Phi_{kj}, \tag{2.51}$$

where δ_{ij} is the Kronnecker delta (2.42). Equivalently, (2.51) can be written in matrix notation as $\det(\Phi)I = C\Phi^T$. Finally, note that the only term in $\det(\Phi)$ that contains a specific element Φ_{ij} is the term $c_{ij}\Phi_{ij}$, so that

$$\frac{\partial}{\partial\Phi_{ij}}\det(\Phi) = c_{ij}. \tag{2.52}$$

Using (2.46), (2.51), (2.52), and the chain rule, the time derivative of the fundamental matrix is

$$\begin{aligned}\frac{d}{dt}\det(\Phi(t)) &= \sum_{i,j=1}^{n} c_{ij}(t)\frac{d}{dt}\Phi_{ij}(t) = \sum_{i,j,k=1}^{n} c_{ij}(t)a_{ik}(t)\Phi_{kj}(t)\\ &= \sum_{i,k=1}^{n} a_{ik}(t)\left(\sum_{j=1}^{n} c_{ij}(t)\Phi_{kj}(t)\right) = \sum_{i,k=1}^{n} a_{ik}(t)\delta_{ik}\det(\Phi(t)).\end{aligned}$$

Simplifying yields

$$\frac{d}{dt}\det(\Phi(t)) = \left(\sum_{i,k=1}^{n}\delta_{ik}a_{ik}(t)\right)\det(\Phi(t)) = \operatorname{tr}(A(t))\det(\Phi(t)).$$

This scalar differential equation for the determinant of Φ can be easily integrated to time t to obtain the promised (2.50). □

Since $\det(\Phi(T,0)) = \det(M)$, M is nonsingular. Consequently, all the Floquet multipliers are nonzero and the Floquet exponents are well defined. Abel's theorem will be used in §4.11 and in §7.2 to aid the study of the stability of periodic and aperiodic orbits.

In addition to the Floquet exponents, $\ln \mu_j$, it is also convenient to define the logarithm of the Floquet matrix, $\ln M$, itself. However, it is not obvious that the logarithm of a general matrix is always well defined, as is the case for the exponential. Since the MacLaurin series defined $\exp(M)$, it would be reasonable to use a similar series for the logarithm,

$$\ln(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}; \tag{2.53}$$

however, this converges only for $|x| < 1$. Since $\ln M = \ln(I-(I-M))$, we assume the series definition can be used only for $\|I-M\| < 1$. How can we define $\ln M$ in general?

Lemma 2.35. *Any nonsingular matrix A has a (possibly complex) logarithm*

$$\ln A = P\ln(\Lambda)P^{-1} - \sum_{j=1}^{n-1} \frac{(-S^{-1}N)^j}{j},$$

where $A = S+N$, $S = P\Lambda P^{-1}$ is semisimple, N is nilpotent, Λ is the diagonal matrix of eigenvalues, and P is the matrix of generalized eigenvectors of A.

Proof. The semisimple-nilpotent decomposition, Theorem 2.23, gives $A = S+N$, where S is semisimple, N is nilpotent, and $[S,N] = 0$. Since A is assumed nonsingular, S is also nonsingular since its eigenvalues are the same as those of A.

Consider first the case of a semisimple, nonsingular matrix S. By definition there exists a diagonalizing transformation P such that $P^{-1}SP = \Lambda$, where Λ is diagonal and has all entries nonzero but is possibly complex. Defining $\ln\Lambda \equiv \operatorname{diag}(\ln\Lambda_{ii})$, then $e^{\ln\Lambda} = \Lambda$, and

$$S = Pe^{\ln\Lambda}P^{-1} = \exp\left(P\ln\Lambda P^{-1}\right), \tag{2.54}$$

so that $\ln S \equiv P\ln\Lambda P^{-1}$. Hence $\ln S$ exists for any nonsingular, semisimple S.

Now suppose that N is any nilpotent matrix. We claim that $\ln(I+N)$ exists. Indeed, using the series (2.53) formally (ignoring convergence), define a matrix B by

$$B = -\sum_{j=1}^{\infty}\frac{(-N)^j}{j} = -\sum_{j=1}^{n-1}\frac{(-N)^j}{j}. \tag{2.55}$$

This is more than a formal definition, however, because, when N is nilpotent, only finitely many terms in this series are nonzero; consequently, (2.55) converges for any N. Moreover we claim that $e^B = I+N$. Formal manipulation of the power series gives

$$e^B = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\sum_{j=1}^{\infty}\frac{(-N)^j}{j}\right)^k = I+N$$

because this is true for scalar values, and $\left[N^j, N^k\right] = 0$ for any integers j and k. Moreover these series converge because the exponential series converges for any bounded linear operator, and the inner series has only finitely many nonzero terms. In conclusion, $B = \ln(I+N)$ is given by (2.55) for any nilpotent N.

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Since $\det(\Phi(T,0)) = \det(M)$, M is nonsingular. Consequently, all the Floquet multipliers are nonzero and the Floquet exponents are well defined. Abel's theorem will be used in §4.11 and in §7.2 to aid the study of the stability of periodic and aperiodic orbits.

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$$\ln(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}; \tag{2.53}$$

however, this converges only for $|x| < 1$. Since $\ln M = \ln(I-(I-M))$, we assume the series definition can be used only for $\|I-M\| < 1$. How can we define $\ln M$ in general?

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because this is true for scalar values, and $\left[N^j, N^k\right] = 0$ for any integers j and k. Moreover these series converge because the exponential series converges for any bounded linear operator, and the inner series has only finitely many nonzero terms. In conclusion, $B = \ln(I+N)$ is given by (2.55) for any nilpotent N.

Finally, consider the general case:

$$A = S + N = S(I + S^{-1}N).$$

Note that since N is nilpotent and $[S,N]=0$, then $S^{-1}N$ is also nilpotent: if $N^k = 0$, then $\left(S^{-1}N\right)^k = S^{-k}N^k = 0$. Therefore, both terms, S and $(I+S^{-1}N)$, have logarithms. By analogy with the property $\ln(ab) = \ln a + \ln b$, we claim that $\ln A$ is given by

$$B = \ln S + \ln(I + S^{-1}N),$$

where the first term is given by (2.54) and the second by (2.55) with $N \to S^{-1}N$. Note that $\left[S, I+S^{-1}N\right] = 0$, and so by their definitions, $\left[\ln S, \ln(I+S^{-1}N)\right] = 0$ as well. This implies that

$$e^B = e^{\ln S + \ln(I+S^{-1}N)} = e^{\ln S} e^{\ln(I+S^{-1}N)} = S(I+S^{-1}N) = A,$$

as claimed. □

Although $\ln A$ exists, it is not unique. Indeed, just as for a scalar, where the exponential of $\ln(a)+2n\pi i$ is independent of $n \in \mathbb{Z}$, the eigenvalues of $\ln A$ are unique only up to addition of $2n\pi i$ (see Exercise 13d).

The definition of $\ln M$ can be used to obtain a nice form for the solutions to a periodic linear system.

Theorem 2.36 (Floquet 1883). *Let M be the monodromy matrix for a T-periodic linear system $\dot{x} = A(t)x$ and $TB = \ln M$ its logarithm. Then there exists a T-periodic matrix $\mathcal{P}$ such that the fundamental matrix solution is*

$$\Phi(t,0) = \mathcal{P}(t)e^{tB}. \tag{2.56}$$

Proof. Let $\Psi(t) = \Phi(t+T,0)$. Since $A(t)$ is periodic, then $\frac{d}{dt}\Psi = A(t+T)\Psi = A(t)\Psi$, with $\Psi(0) = M$. Now since Φ is the fundamental matrix solution, every solution $x(t)$ is of the form $\Phi(t,0)x(0)$; accordingly $\Psi(t) = \Phi(t,0)M$, and

$$\Phi(t+T,0) = \Phi(t,0)M = \Phi(t,0)e^{TB}.$$

Since e^{tB} is nonsingular, define $\mathcal{P}(t) \equiv \Phi(t,0)e^{-tB}$ so that

$$\mathcal{P}(t+T) = \Phi(t+T,0)e^{-(t+T)B} = \Phi(t,0)e^{TB}e^{-(t+T)B} = \mathcal{P}(t).$$

Therefore, $\mathcal{P}$ is T-periodic. □

As usual, it is not always satisfactory to write the solution of a real linear system in terms of complex functions. However, at the expense of doubling the period, a real form can be found, as follows.

Theorem 2.37. *Let Φ be the fundamental matrix solution for the time T-periodic linear system (2.45). Then there exist a real $2T$-periodic matrix $\mathcal{Q}$ and real matrix R such that*

$$\Phi(t,0) = \mathcal{Q}(t)e^{tR}.$$

Proof. In Exercise 21, you will show that for any nonsingular matrix M, there exists a real matrix R such that $M^2 = e^{2TR}$. Define $\mathcal{Q}(t) = \Phi(t,0)e^{-tR}$, and then

$$\mathcal{Q}(t+2T) = \Phi(t+2T,0)e^{-2TR}e^{-tR} = \Phi(t,0)M^2M^{-2}e^{-tR} = \mathcal{Q}(t).$$

Therefore, $\mathcal{Q}$ is $2T$-periodic. □

In fact, one need only extend the period to $2T$ when M has negative real multipliers (see Exercise 21). These, as we will see later in Chapter 8, typically arise near a "period-doubling bifurcation."

2.9 ▪ Exercises

You should do these problems by hand; however, feel free to use a computer to check your answers if that is possible.

1. Near an equilibrium an ODE can be simplified by expanding the equations to first order in the deviations of the variables from their equilibrium values. The resulting system is linear. Formally for $\dot{x} = f(x)$, set $x = x_{eq} + \delta x$, and use $f(x_{eq}) = 0$ to find

$$\delta \dot{x} = f(x_{eq} + \delta x) \approx f(x_{eq}) + \frac{\partial f}{\partial x}(x_{eq})\delta x + \cdots \approx A\delta x.$$

   Here you must remember that x is a vector, and so the matrix A has elements $a_{ij} = \partial f_i / \partial x_j$. Carry out this expansion for the equilibria you found in Exercise 1.2 and compute the 4×4 matrix A for each case.

2. Find the general solution to the two-dimensional linear system for the Hamiltonian (1.29) and show that the phase portrait given in Figure 1.8 is correct.

3. Show that if T is a bounded linear operator and is invertible, then

$$\left\| T^{-1} \right\| \geq \frac{1}{\|T\|}.$$

4. Suppose T is a bounded linear operator on X that leaves a complete, vector subspace $E \subset X$ invariant (i.e., whenever $v \in E$ then $T(v) \in E$). Show that e^T also leaves E invariant. (For the definition of complete, normed space, see Sec. 3.2.)

5. In this problem we will prove the following lemma.

   Lemma 2.38. *A linear operator T is bounded if and only if it is continuous.*

   (a) Recall that continuity means that if $x_n \to x$, then $T(x_n) \to T(x)$. First show that linearity implies that if T is continuous at $x = 0$, then it is continuous everywhere. (*Hint*: Consider a sequence $x_n \to 0$ and then use superposition to find the limit of $T(x_n + y)$.)

   (b) Suppose T is bounded; then show that $x_n \to 0$ implies that $|T(x_n)| \to 0$. Argue that this implies T is continuous.

   (c) Suppose T is not bounded; then show that it is not continuous at $x = 0$. (*Hint*: Argue that there is sequence x_n such that $|T(x_n)| > n|x_n|$. Now let $y_n = x_n / n|x_n|$). Argue that you have proved that if T is continuous, it is bounded.