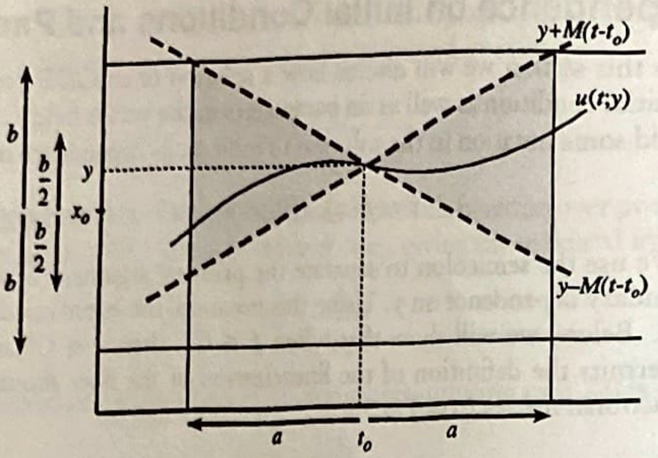

Figure 3.6. *Existence of solutions for initial conditions in a neighborhood of radius b about x_o requires using a smaller ball.*

Note that the initial conditions can be varied only over a ball with *half* the radius of the ball where f is assumed to be nice and that the solution can be shown to exist only for *half* of the time. This is because all the solutions must stay in B_b for all $|t| < a$; see Figure 3.6. We could adjust these factors of ½, increasing one at the expense of decreasing the other. Finally, as before, the requirement that $a < 1/K$ could be eliminated with a little more work.

Example 3.27. Consider the initial value problem (3.23) taking as the central point $x_o = 0$ so that $f : B_b(0) \to \mathbb{R}$. The Lipschitz constant on this domain is $K = 2b$ and $|f|$ is bounded by $M = b^2$. The theorem then guarantees that a unique solution exists for $|y| < b/2$, providing $a < \min\{(2b)^{-1}, b/(2b^2)\} = (2b)^{-1}$. Note that the actual solution (3.25) for an initial condition $y \in B_{b/2}(0)$ blows up a time $t = 1/y$, so the shortest time occurs when $y = b/2$. Thus the true solution exists at least four times longer than the theorem gives us. ∎

So far we have seen that the solution $u(t;y)$ exists for a range of initial conditions and is C^1 in t whenever the vector field f is Lipschitz. Our goal now is to discuss the smoothness of the dependence of $u(t;y)$ on y. For example, we will see that when the vector field is Lipschitz, u is a Lipschitz function of y.

The main tool used to prove this is a lemma about *differential inequalities*. Some care must be exercised here. For example, suppose that $f < g$; does it follow that $\dot{f} < \dot{g}$? A simple counterexample shows this is not true: $f(t) = \cos 3t$ and $g(t) = 2$. The converse statement is also not true: for example, if $f(t) = \sin t$ and $g(t) = 2t$, then indeed $\dot{f}(t) = \cos t < \dot{g}(t) = 2$, but note that $f > g$ when $t < 0$. In contrast, note that if $\dot{f} \le \dot{g}$, it follows that f increases less rapidly than g, so that $f(t) - f(t_o) \le g(t) - g(t_o)$ *provided* $t \ge t_o$. It is important, of course, that we assume that both $f, g \in C^1$ for this to work. This simple idea leads to the lemma proved by Thomas Grönwall in 1919.

Lemma 3.28 (Grönwall). *Suppose $g, k : [0,a] \to \mathbb{R}$ are continuous, $a > 0$, $k(t) \ge 0$,* *and g obeys the inequality*

$$g(t) \le G(t) \equiv c + \int_0^t k(s)g(s)ds \tag{3.31}$$

for all $0 \le t \le a$. *Then for all* $t \in [0,a]$,

$$g(t) \le c e^{\int_0^t k(s)ds}. \tag{3.32}$$

Proof. Since g and k are continuous, then G is C^1 and $G(0) = c$. Differentiation of G from (3.31) gives

$$\dot{G}(t) = k(t)g(t) \le k(t)G(t);$$

consequently, $\dot{G} - kG \le 0$. Multiplying by the positive "integrating factor" $e^{-\int_0^t k(s)ds}$ gives

$$e^{-\int_0^t k(s)ds}(\dot{G}(t) - kG) = \frac{d}{dt}\left(G(t)e^{-\int_0^t k(s)ds}\right) \le 0.$$

Integrating this inequality finally implies

$$G(t)e^{-\int_0^t k(s)ds} \le G(0) \Rightarrow G(t) \le c e^{\int_0^t k(s)ds}.$$

Since $g \le G$, we obtain (3.32). □

A similar lemma holds when c is allowed to be a function of time—see Exercise 11.

Grönwall's inequality makes the proof of our desired theorem very easy.

Theorem 3.29 (Lipschitz Dependence on Initial Conditions). *Let* $x_o \in \mathbb{R}^n$, *and suppose there is a* $b > 0$ *such that* $f : B_b(x_o) \to \mathbb{R}^n$ *is Lipschitz with constant K and that* $J = [-a,a]$ *is the common interval of existence for solutions* $u : J \times B_{b/2}(x_o) \to B_b(x_o)$. *Then* $u(t;y)$ *is uniformly Lipschitz in* y *with Lipschitz constant* e^{Ka}.

Proof. Suppose $u(t;y)$ and $u(t;z)$ are two solutions starting in $B_{b/2}(x_o)$. They have a common interval of existence J. When $t \in [0,a]$, the integral form (3.11) implies that

$$|u(t;y) - u(t;z)| \le |y - z| + \int_0^t |f(u(\tau;y)) - f(u(\tau;z))|\, d\tau$$
$$\le |y - z| + K\int_0^t |u(\tau;y) - u(\tau;z)|\, d\tau.$$

This is precisely Grönwall's form (3.31) with $c = |y - z|$, and $k(t) = K$, so (3.32) becomes

$$|u(t;y) - u(t;z)| \le |y - z| e^{Kt}. \tag{3.33}$$

A similar inequality holds for $t \in [-a, 0]$, giving our result. □

A slightly different proof is sketched in Exercise 10.

We can use this Lipschitz dependence of $u(t;y)$ on y to prove that when f is C^1, then u is also C^1 in y. The proof of this result requires a bit more work than the previous one.