Homework #1, Due Date: 01/23/26
MTH 752: Spring 2026
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Problem 1. Consider the following dynamical system on the torus $(\theta_1, \theta_2) \in S^1 \times S^1$:

$$\dot{\theta}_1 = \omega_1 + \sin(\theta_1)\cos(\theta_2),$$
$$\dot{\theta}_2 = \omega_2 + \sin(\theta_2)\cos(\theta_1),$$

where $\omega_1, \omega_2 \geq 0$. On the unit square representation of the torus, sketch all the qualitatively different phase portraits that can arise as ω_1 and ω_2 vary.

Problem 2. Consider the flow on the torus given by

$$\dot{\theta}_1 = \omega_1,$$
$$\dot{\theta}_2 = \omega_2,$$

where $\omega_1, \omega_2 > 0$ and ω_1/ω_2 is irrational. A trajectory is called **dense** on the torus if given any $\mathbf{p} \in S^1 \times S^1$ and any $\varepsilon > 0$, there is some $t^*(\varepsilon) < \infty$ such that the trajectory satisfies $\|(\theta_1(t^*), \theta_2(t^*)) - \mathbf{p}\| < \varepsilon$. Prove that each trajectory for this system is **dense** on the torus.

Problem 3. Consider the following dynamical system on the torus $(\theta_1, \theta_2) \in S^1 \times S^1$:

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1),$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2),$$

where $\omega_1, \omega_2, K_1, K_2 > 0$.

(a) Prove that the system has no fixed points.

(b) Find a conserved quantity for the system. **Hint:** You can use the differential equations to express the quantity $\sin(\theta_2 - \theta_1)$ in two different ways. Equating and differentiating with respect to time can be helpful.

(c) The **winding number** of a periodic or quasiperiodic orbit on the torus is given by $\lim_{t\to\infty} \theta_1(t)/\theta_2(t)$ and the **long-time** average of a quantity $f(t)$ is defined by

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt.$$

Assuming $K_1 = K_2$, for this system evaluate the long term averages $\langle \dot{\theta}_1 + \dot{\theta}_2 \rangle$ and $\langle \dot{\theta}_1 - \dot{\theta}_2 \rangle$ and use this to compute the winding number of any orbit.

Problem 4. Consider the vector field on the cylinder $(\theta, y) \in S^1 \times \mathbb{R}$ given by

$$\begin{aligned} \dot{\theta} &= 1, \\ \dot{y} &= ay, \end{aligned}$$

where $a \in \mathbb{R}$. Define an appropriate Poincare map and find a formula for it. Show that the system has a periodic orbit and determine its stability for all $-\infty < a < \infty$.

Problem 5. Prove that the *non-autonomous* system $\dot{x} + \sin(x) = \sin(t)$ has at least two periodic solutions. **Hint:** Define an appropriate Poincare map, and regard the system as a vector field on the cylinder $(t, x) \in S^1 \times \mathbb{R}$ by letting $\dot{t} = 1$ and $\dot{x} = \sin(t) - \sin(x)$.