Homework #1, Due Date: 02/06/26
MTH 752: Spring 2026
Instructor: John Gemmer

In these homework assignment, we will study properties of the Lorenz equations:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{aligned}$$

where $\sigma, r, b > 0$.

Problem 1. For the Lorenz equations, show that the characteristic polynomial for the Jacobian matrix at C^+, C^- is

$$p(\lambda) = \lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1).$$

By seeking roots of $p(\lambda)$ of the form $\lambda = i\omega$, where ω is real, show that there is a pair of pure imaginary eigenvalues at the Hopf bifurcation point:

$$r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right).$$

Problem 2. Show that there is an ellipsoidal region E of the form

$$rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C$$

such that all trajectories of the Lorenz equations eventually enter E and stay in there forever.

Problem 3. Show that the z-axis is an invariant line for the Lorenz equations.

Problem 4. Consider the following system in polar coordinates

$$\begin{aligned} \dot{r} &= r(1 - r^2), \\ \dot{\theta} &= 1. \end{aligned}$$

Let D be the closed unit disk $x^2 + y^2 \leq 1$.

(a) Is D an invariant set? Explain your answer.

(b) Does D attract an open set of initial conditions? Explain your answer.

(c) Is D an attractor? If not why not? If so, find its basin of attraction.

(d) Repeat part (c) for the unit circle $x^2 + y^2 = 1$.

Problem 5. Consider the map

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \frac{1}{2}, \\ 2 - 2x_n, & \frac{1}{2} \le x_n \le 1, \end{cases}$$

as a simple analytical model of the Lorenz map.

(a) Why is it called the "tent map"?

(b) Find all the fixed points and classify their stability.

(c) Show that the map has a period-2 orbit. Is it stable or unstable?

(d) Can you find any period-3 orbits? How about period 4? If so, are the corresponding orbits stable or unstable?