Homework #3, Due Date: 02/23/26
MTH 752: Spring 2026
Instructor: John Gemmer

Problem 1. Suppose you want to find the roots of an equation $g(x) = 0$ where $g(x)$ is a smooth function. In Calculus, you might have learned Newton's method which says that given an initial guess of x_0, the iterations

$$x_{n+1} = f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}$$

form a sequence of approximations of a root.

1. Show that in most cases if x^* is a root of $g(x)$ then it is a fixed point of this iterated map. Under what conditions is this statement not true?

2. A fixed point of an iterated map $x_{n+1} = f(x_n)$ is called **superstable** if $f'(x^*) = 0$. Show that in almost all cases the roots of $g(x)$ are superstable fixed points of this iterated map.

3. Suppose x^* is a root of $g(x)$ and $f''(x^*) \neq 0$. Show that there is an interval $[a, b]$ containing x^* and a constant $M > 0$ such that if $x_0 \in [a, b]$ then $|f(x_0) - x^*| = |x_1 - x^*| < M|x_0 - x^*|^2$. **Hint:** Taylor's theorem with remainder could be useful.

4. Suppose x^* is a root of $g(x)$ and $f''(x^*) \neq 0$. Using part (c), show that there is an interval $[a, b]$ containing x^* and a constant $M > 0$ such that if $x_0 \in [a, b]$ then $|x_n - x^*| < (M|x_0 - x^*|)^{2^n}$.

5. Given the results in part (d), what conditions on M and x_0 guarantee that the iterations will converge to the root x^*.

6. If the conditions in part (e) are satisfied, and $|x_0 - x^*| < 10^{-1}$, how many iterations would you roughly have to compute in order for the error to satisfy $|x_n - x_0| < 10^{-16}$. **Note:** This rapid rate of convergence motivates why we call these fixed points superstable.

Problem 2. The decimal shift map on the unit interval $[0, 1]$ is given by

$$x_{n+1} = 10x_n \bmod 1.$$

(a) Find all the fixed points for this system.

(b) Show that the map has periodic orbits of all periods, but that all of them are unstable. For the first part, it is sufficient to explain how you would construct a period p orbit, for each integer $p > 1$.

(c) Show that this map has infinitely many distinct orbits that are not periodic or fixed points.

(d) Show that the system exhibits sensitive dependence on initial conditions.

(e) Explicitly construct a dense orbit for this system. **Hint:** My statement in class of just pick any irrational number was wrong. First for $n \in \mathbb{N}$ consider how you would use a finite decimal expansion to construct an orbit that will be a distance less than 10^{-n} between every point in $[0, 1]$. Second, glue all of these finite decimal expansions together to create an infinite decimal expansion whose orbit will be dense.

(f) An "eventually fixed point" of an iterated map is a point that iterates to a fixed point after a finite number of steps; thus $x_{n+1} = x_n$ for all $n > N$ where $N \in \mathbb{N}$. Is the number of eventually-fixed points for the decimal shift map countable or uncountable?

Problem 3. Analyze the long term behavior of the iterated map

$$x_{n+1} = r\frac{x_n}{1+x_n^2},$$

where $r > 0$. Specifically, find and classify the stability of all fixed points as a function of r. Can there be periodic orbits for this iterated map? Can this iterated map be chaotic?

Problem 4. If n is an odd number, the "even-n fold odd Cantor set" is constructed as follows. First, construct S_1 by dividing $[0,1]$ into n intervals of equal length and delete all the even pieces, i.e., the second, fourth, sixth, intervals etc. Second, construct S_2 by replicating the construction of S_1 on each subinterval of S_1. Continue this process inductively to construct similar sets S_i for $i \in \mathbb{N}$. The even-n fold odd Cantor set is then $S = \cap_{i=1}^{\infty} S_i$. **Note:** The classic Cantor set introduce in class is the even-3 fold odd Cantor set.

(a) Prove for all $n \in \mathbb{N}$ that S_∞ is uncountable.

(b) Prove for all $n \in \mathbb{N}$ that S_∞ is a set of measure zero.

(c) Find the box dimension of S_∞ as a function of n.

Problem 5. Let $r > 2$ and consider the following tent map on $[0,1] \cup \infty$ defined by $x_{n+1} = f(x_n)$, where

$$f(x) = \begin{cases} rx & 0 \le x \le \frac{1}{2} \\ r(1-x) & \frac{1}{2} < x \le 1 \\ \infty & x = \infty \end{cases},$$

with the understanding that any point that is mapped outside the interval $[0,1]$ is mapped to ∞. For example, if $r = 4$ and $x_0 = \frac{1}{3}$ then $f(x_0) = \infty$. We say that an initial condition has escaped after k iterations if $x_k = \infty$ but $x_{k-1} \neq \infty$.

(a) Find the set of initial conditions x_0 that escape after one or two iterations.

(b) Describe the set of x_0 that never escape.

(c) Find the box dimension of the set that never escape. **Note:** This set is an invariant set.

(d) Show that the Lyapunov exponent is positive at each point in this invariant set.