Homework #5, Due Date: 04/10/26
MTH 752: Spring 2026
Instructor: John Gemmer

Problem 1. For a matrix $A \in \mathbb{R}^{n\times n}$ the Euclidean matrix norm is defined by

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|,$$

where the norms on the right hand side are the standard Euclidean norms of vectors.

(a) If $I \in \mathbb{R}^{n\times n}$ is the standard identity matrix, prove that $\|I\| = 1$.

(b) Prove for all $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$ that $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$.

(c) Prove that if $A, B \in \mathbb{R}^{n\times n}$ then $\|AB\| \leq \|A\|\|B\|$.

(d) For an invertible matrix $A \in \mathbb{R}^{n\times n}$ the condition number of a matrix is defined by

$$\text{cond}(A) = \|A\|\|A^{-1}\|.$$

Prove that for an invertible matrix $A \in \mathbb{R}^{n\times n}$, $\text{cond}(A) \geq 1$.

Problem 2. Reviewing Marissa's lecture could be helpful for this problem. Consider the linear initial value problem

$$\dot{\mathbf{x}} = A(t)\mathbf{x},$$
$$\mathbf{x}(0) = \mathbf{x}_0,$$

where $A : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ is an $n \times n$ continuous matrix valued function satisfying

$$\|A(t)\| \leq M,$$

for all $t \in \mathbb{R}$ and some $M > 0$.

(a) Show that solutions to this differential equation satisfy

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}_0\| + \int_0^t M\|\mathbf{x}(s)\|ds.$$

(b) Show that

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}_0\|e^{Mt}.$$

Note: You cannot simply state this is true by Gronwall's inequality. I am essentially asking you to show Gronwall's inequality.

Problem 3. Consider the following boundary value problem

$$\begin{cases} \varepsilon y'' + y' = 2x \\ y(0) = 0 \text{ and } y(1) = 0 \end{cases}.$$

Determine the lowest order outer and inner solutions to the problem that correctly satisfy the matching condition and combine them to find the lowest order composite solution to this boundary value problem.

Problem 4. Consider the following boundary value problem

$$\begin{cases} \varepsilon y'' - (2x+1)y' + y^2 = 0 \\ y(0) = 0 \text{ and } y(1) = 1 \end{cases}.$$

Determine the lowest order outer and inner solutions to the problem that correctly satisfy the matching condition and combine them to find the lowest order composite solution to this boundary value problem.

Problem 5. Consider the following boundary value problem

$$\begin{cases} \varepsilon y'' - \varepsilon y y' - y = x \\ y(0) = 2 \text{ and } y(1) = 0 \end{cases}.$$

Determine the lowest order outer and inner solutions to the problem that correctly satisfy the matching condition and combine them to find the lowest order composite solution to this boundary value problem.

Problem 6. This problem walks you through a typical approach for using integrals to compute pointwise information about a function. This type of analysis is crucial in the calculus of variations. Our goal is to prove that if $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous then

$$\lim_{\Delta x \to 0^+} \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(s)ds = f(x).$$

Recall that a function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous at a point x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

(a) Show that

$$\frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(s)ds - f(x) = \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} (f(s) - f(x))ds.$$

(b) Show that for all $\varepsilon > 0$ that if Δx is sufficiently small then

$$\left| \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(s)ds - f(x) \right| \leq \varepsilon.$$

and thus deduce that

$$\lim_{\Delta x \to 0} \left| \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(s)ds - f(x) \right| = 0$$

which implies that

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(s)ds = f(x).$$

Problem 7. Suppose η is a positive smooth function compactly supported on $[-1,1]$ that satisfies

$$\int_{-\infty}^{\infty} \eta(x)dx = \int_{-1}^{1} \eta(x)dx = 1$$

and define the sequence of functions $\eta_\varepsilon(x) = \varepsilon^{-1}\eta\left(\varepsilon^{-1}x\right)$.

(a) Determine the support of $\eta_\varepsilon(x)$ and sketch characteristic graphs of $\eta_\varepsilon(x)$ as $\varepsilon \to 0$.

(b) Prove that

$$\int_{-\infty}^{\infty} \eta_\varepsilon(x)dx = 1.$$

(c) Prove that if f is a continuous function then

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(x)\eta_\varepsilon(x)dx = f(0).$$

Hint: This proof mimics many of the same ideas that were used in problem #6.

(d) Prove that if f is a continuous function then

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(x)\eta_\varepsilon(x-s)dx = f(s).$$

Problem 8. Suppose that $f : [0,1] \mapsto \mathbb{R}$ is a continuous function satisfying

$$\int_{0}^{1} f(x)\eta(x)dx = 0$$

for all smooth functions with compact support η. Prove that f is identically zero on $[0,1]$.

Problem 9. A dynamical system in the phase-space $\mathbb{R}^n \times \mathbb{R}^n$ is called a Hamiltonian system if there exists a smooth function $H(\mathbf{x}, \mathbf{p})$ from $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$, called a Hamiltonian, such that the components of $\mathbf{x}$ and $\mathbf{p}$ satisfy

$$\dot{x}_i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

The vector $\mathbf{x} = (x_1, \ldots, x_n)$ is called the generalized coordinates and $\mathbf{p} = (p_1, \ldots, p_n)$ is called the generalized momentum. Using what Jack demonstrated in his presentation, show that the volume of closed subset of $\mathbb{R}^n \times \mathbb{R}^n$ is preserved under the dynamics of this system.