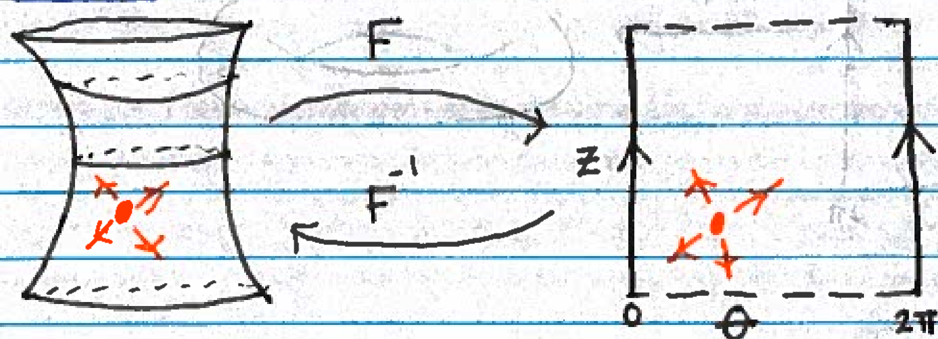
Lecture #1: Motion on Tori and Cylinders

Manifolds



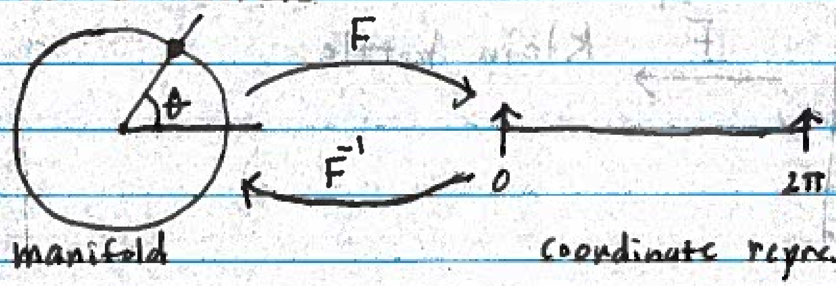
↑ → Means sides are identified, i.e. they are the sam

M = hyperboloid with a vector field.

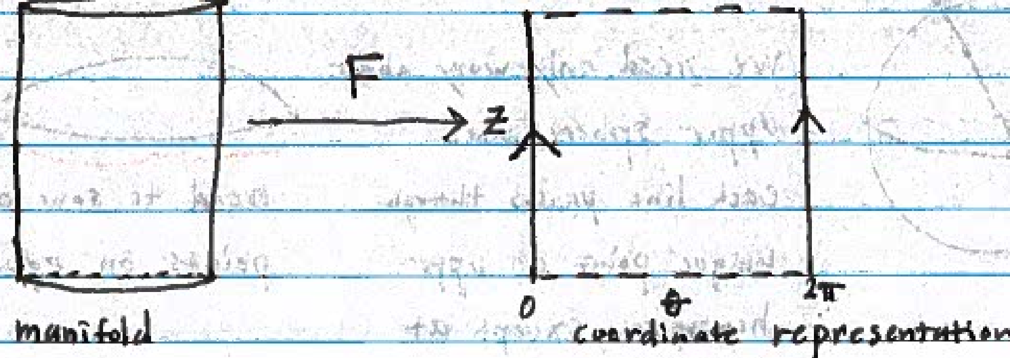
How do we describe flow on M?? Key idea is to create maps F which are diffeomorphisms into $\mathbb{R}^n$ on which calculus can be done. If this is possible, we call M an n-dimensional manifold and the collection of all maps is an atlas.

Examples:

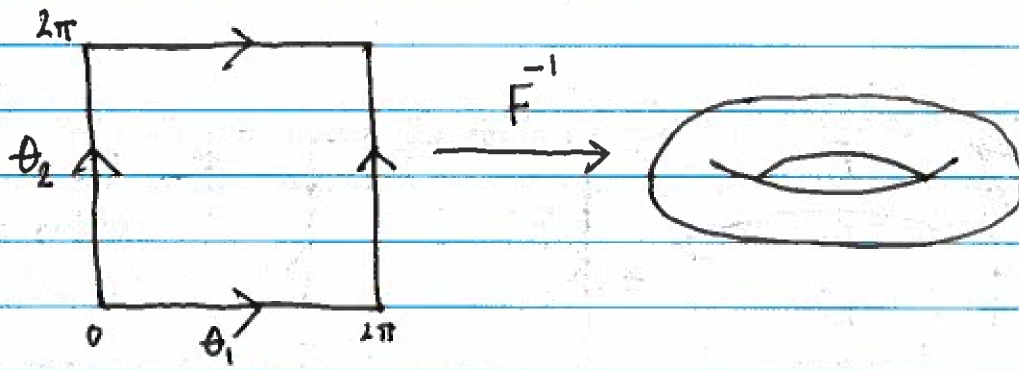
1. $M = S^1 = \text{circle}$



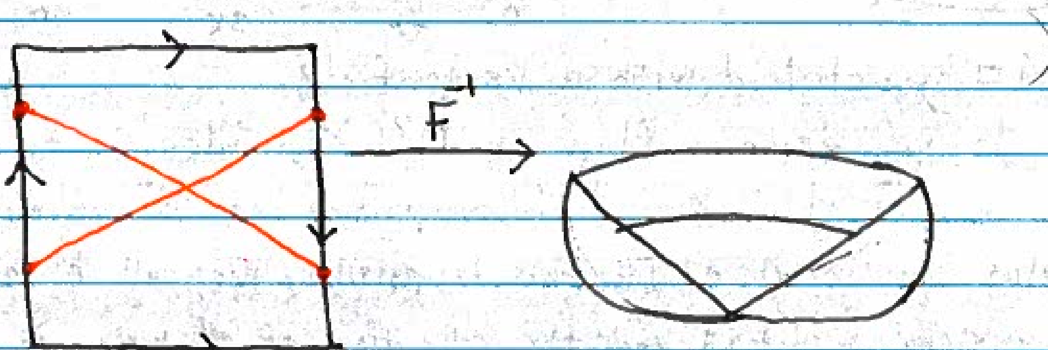
2. $M = S^1 \times \mathbb{R} = \text{cylinder}$



3. $M = S^1 \times S^1 = $ torus.



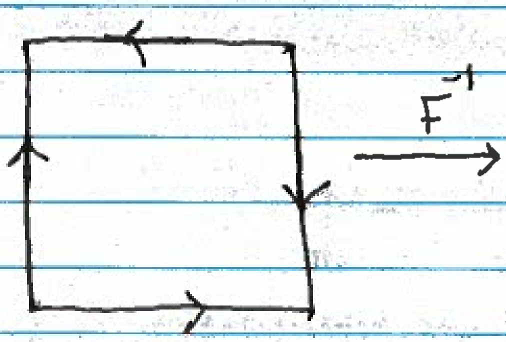
4.



Möbius strip

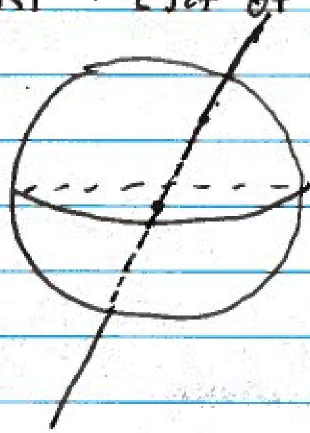
Dynamics difficult to interperet

5.



Klein bottle.

6. $\mathbb{R}P^2 = \{$set of lines through origin in $\mathbb{R}^3\}$.



We need only worry about upper sphere since each line passes through unique point on upper hemisphere, except at equator

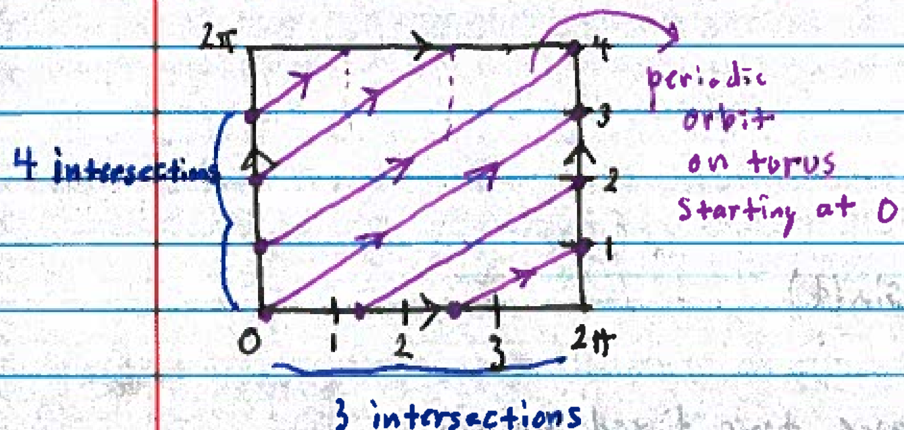
need to sew antipodal points on equator.

Motion on a torus

Example:

$$\dot{\theta}_1 = 4 \quad , \quad (\theta_1, \theta_2) \in S^1 \times S^1$$
$$\dot{\theta}_2 = 3$$

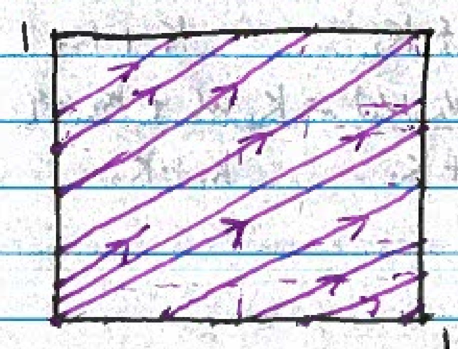
$$\frac{d\theta_2}{d\theta_1} = \frac{3}{4} \Rightarrow \theta_2(\theta_1) \text{ is a line of slope } 3/4$$



This is called a 3:4 resonant tori. The winding number is $3/4$.

Example:

$$\dot{\theta}_1 = \sqrt{2} \Rightarrow \frac{d\theta_2}{d\theta_1} = \frac{1}{\sqrt{2}}$$
$$\dot{\theta}_2 = 1$$



This is called a nonresonant tori. The orbit is called quasiperiodic.

Example:

$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$, $(\theta_1, \theta_2) \in S^1 \times S^1$

$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$

two joggers on a track
trying to match speed

$\omega_i \sim$ natural speed / frequency

$K_i \sim$ coupling strength

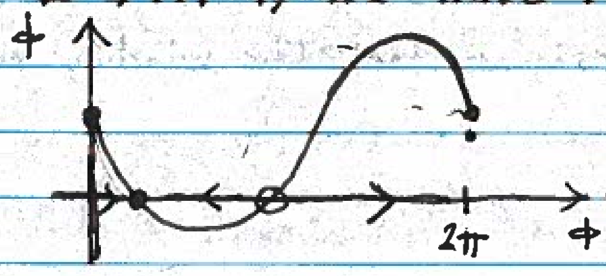
Let $\phi = \theta_2 - \theta_1$:

$\dot{\phi} = \dot{\theta}_2 - \dot{\theta}_1$

$\Rightarrow \dot{\phi} = \omega_2 - \omega_1 - (K_1 + K_2)\sin(\phi)$

$\Rightarrow \dot{\phi} = \Omega - K\sin(\phi)$,

If $|\Omega/K| < 1$, we have two fixed points



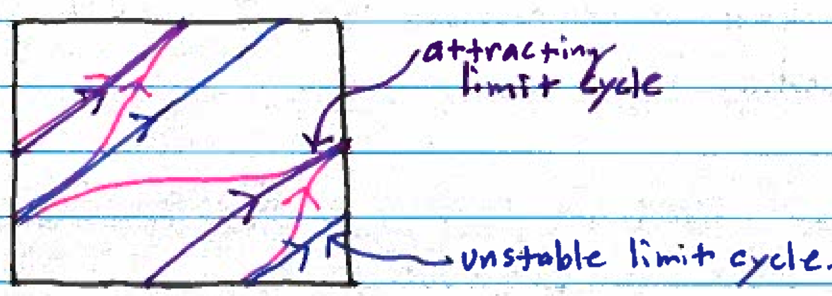
When $\sin(\phi) = \Omega/K$ we have:

$$\dot{\theta}_1 = \omega_1 + K_1\left(\Omega/K\right) = \omega_1 + \frac{K_1(\omega_2 - \omega_1)}{K_1 + K_2} = \frac{K_2\omega_1 + K_1\omega_2}{K_1 + K_2}$$

$$\dot{\theta}_2 = \omega_2 + K_2\left(-\Omega/K\right) = \omega_2 - \frac{K_2(\omega_2 - \omega_1)}{K_1 + K_2} = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$$

Therefore, when $\sin(\phi) = \Omega/K$

$$\frac{d\theta_2}{d\theta_1} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = 1$$



Motion on a Cylinder

$\dot{\phi} = y$, $(\phi, y) \in S^1 \times \mathbb{R}$, $I, \alpha > 0$.

$\dot{y} = I - \sin(\phi) - \alpha y$

Equivalent form:

$\ddot{\phi} = I - \sin(\phi) - \alpha \dot{\phi}$

Damped oscillator.

Nullclines satisfy:

$\dot{\phi} = 0 \Rightarrow y = 0$

$\dot{y} = 0 \Rightarrow y = \frac{1}{\alpha}(I - \sin(\phi))$

Case 1 ($I<1$):

Fixed point when $y=0$ and $\sin(\phi) = I$

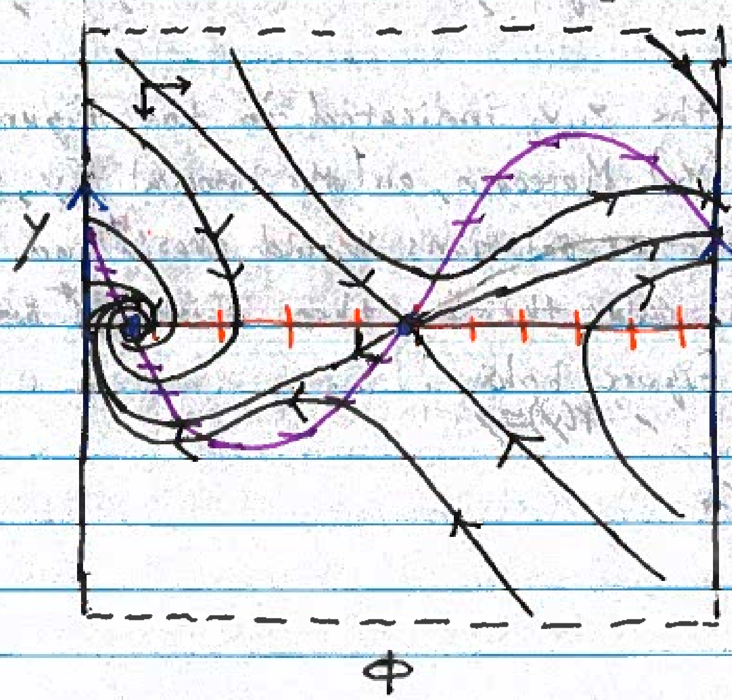
$\Rightarrow (\phi^*, y^*) = (\sin^{-1}(I), 0), (\pi - \sin^{-1}(I), 0)$.

Jacobian is

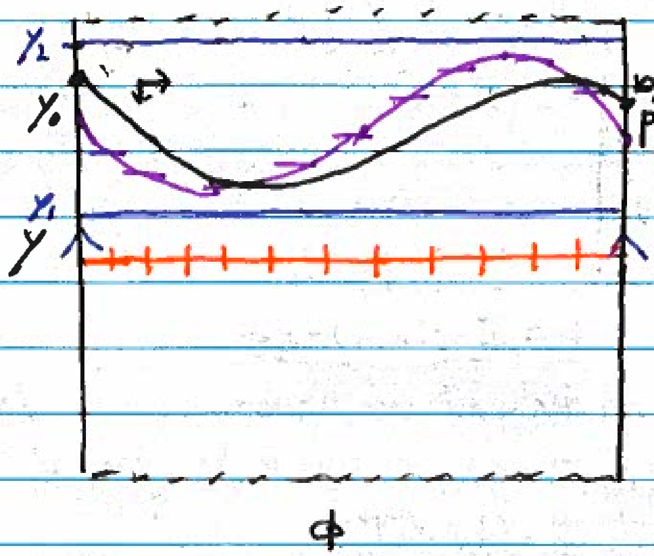
$$J = \begin{bmatrix} 0 & 1 \\ -\cos(\phi) & -\alpha \end{bmatrix}$$

$$J(\sin^{-1}(I), 0) = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-I^2} & -\alpha \end{bmatrix} \Rightarrow \text{Stable node/Stable spiral}$$

$$J(\pi - \sin^{-1}(I), 0) = \begin{bmatrix} 0 & 1 \\ \sqrt{1-I^2} & -\alpha \end{bmatrix} \Rightarrow \text{saddle}$$



Case II ($I>1$):



can this orbit close up on itself into a limit cycle?

Define $P: \mathbb{R} \to \mathbb{R}$, called a Poincaré map, to be the value $P(y)$ after one loop around the cylinder for a trajectory starting at $(0,y)$.

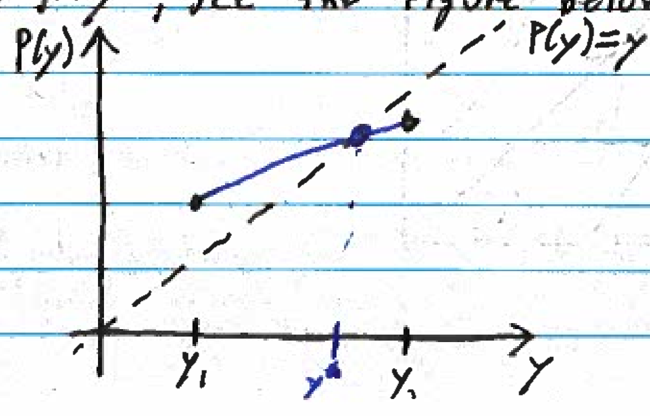
Key idea:

To show that there exists a limit cycle we need to find y^* such that $P(y^*) = y^*$. Such a y^* is called a fixed point.

Theorem- There exists a unique y^* such that $P(y^*) = y^*$.

proof:

By construction, for the y_1, y_2 indicated in the figure above $y_1 < P(y_1)$ and $y_2 > P(y_2)$. Moreover, on the interval $[y_1, y_2]$ $P(y)$ must be monotone or solutions would cross and thus by the intermediate value theorem there exists y^* such that $P(y^*) = y^*$; see the figure below:



Now, suppose there are two limit cycles $y_+(\phi) > y_-(\phi)$. Define

$$E[y(\phi)] = \frac{1}{2}y^2 - \cos(\phi)$$

Along any periodic orbit we have

$$0 = E[y_+(2\pi)] - E[y_+(0)] = E[y_-(2\pi)] - E[y_-(0)]$$

$$\Rightarrow 0 = \int_0^{2\pi} \left.\frac{dE}{d\phi}\right|_{y_+} d\phi = \int_0^{2\pi} \left.\frac{dE}{d\phi}\right|_{y_-} d\phi$$

But, $\frac{dE}{d\phi} = y\frac{dy}{d\phi} + \sin(\phi)$,

$$= y\left(\frac{I - \sin(\phi) - \alpha y}{y}\right) + \sin(\phi),$$

$$= I - \sin(\phi) - \alpha y + \sin(\phi),$$

$$= I - \alpha y.$$

Therefore,

$$\int_0^{2\pi} (I - \alpha y_+(\phi))\, d\phi = \int_0^{2\pi} (I - \alpha y_-(\phi))\, d\phi,$$

$$\Rightarrow \int_0^{2\pi} y_+(\phi)\, d\phi = \int_0^{2\pi} y_-(\phi)\, d\phi,$$

Which is a contradiction.

■

Stability when $I \geq 1$:

A poincare map allows us to think of the dynamical system as an iterated map

$$y_{n+1} = P(y_n), \quad n \in \{0, 1, 2, \dots\}$$

A value y^* is a fixed point of the Poincare map if

$$P(y^*) = y^*.$$

A fixed point y^* is attracting if there exists $\delta > 0$ such that $|y_0 - y^*| < \delta$.

Theorem - If y^* is a fixed point of P and $P'(y^*)=\lambda$ satisfies $|\lambda|<1$ then y^* is attracting. The value λ is called the Floquet multiplier.

proof:

Since $|\lambda|<1$, it follows from continuity of P' that there exists $\delta>0$ such that for all $y\in \eta_\delta(y^*)=\{y : |y-y^*|<\delta\}$, $|P'(y)|\leq M<1$.

Now, if we we let $y_0=y^*+\Delta y\in\eta_\delta(y^*)$ it follows that $|\Delta y|<\delta$ and

$$
\begin{aligned}
|y_1-y^*| &= |P(y_0)-y^*|,\\
&= |P(y^*+\Delta y)-y^*|,\\
&= |P(y^*)+P'(c_1)\Delta y-y^*|, \quad \text{(Taylor's Theorem with Remainder)}\\
&= |y^*+P'(c_1)\Delta y-y^*|,\\
&\leq M|\Delta y|<\Delta y,
\end{aligned}
$$

where $c_1\in\eta_\delta(y^*)$. Thus, $y_1\in\eta_\delta(y^*)$ and

$$
\begin{aligned}
|y_2-y^*| &= |P(y_1)-y^*|,\\
&= |P(P(y_0))-y^*|,\\
&= |P(P(y^*+\Delta y))-y^*|\\
&= |P(P(y^*))+P'(P(c_2))P'(c_2)\Delta y-y^*|,\\
&= |y^*+P'(P(c_2))P'(c_2)\Delta y-y^*|,\\
&\leq M^2|\Delta y|<\Delta y,
\end{aligned}
$$

where $c_2\in\eta_\delta(y^*)$. and thus $y_2\in\eta_\delta(y^*)$. Continuing inductively we have

- $|y_n-y^*|<M^n\Delta y$
- $y_n\in\eta_\delta(y^*)$.

Consequently,

$$\lim_{n\to\infty}|y_n-y^*|=0 \Rightarrow \lim_{n\to\infty}y_n=y^*.$$

∎