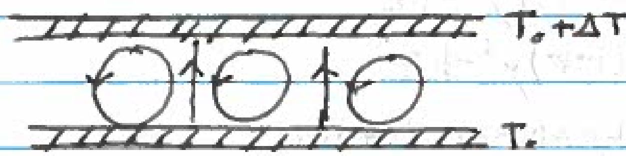
Lecture #2: Lorenz Equations



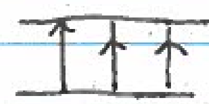
$\dot{x} = \sigma(y-x)$ $\quad\quad$ σ - Prandtl number

$\dot{y} = rx - y - xz$ $\quad\quad$ r - Rayleigh number ($r \ll 1$ conduction, $r \gg 1$ convect

$\dot{z} = xy - bz$ $\quad\quad$ b - dimensionless aspect ratio.

What Happens as r is varied?

Fixed Points:



1. $(0,0,0) \rightarrow$ Pure Conduction
2. $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) \rightarrow$ left and right moving roles

$$\Rightarrow J(x,y,z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$$

$$\Rightarrow J(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \Rightarrow \lambda_{1,2} = -\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\sigma r}$$
$$\lambda_3 = -b$$

Case 1 ($r<1$):

Only 1 fixed and it is a stable node. How can we eliminate closed orbits?

Let $V: \mathbb{R}^3 \rightarrow \mathbb{R}^+$ be defined by

$$V(x,y,z) = \frac{1}{2}\left(\frac{1}{\sigma}x^2 + y^2 + z^2\right)$$

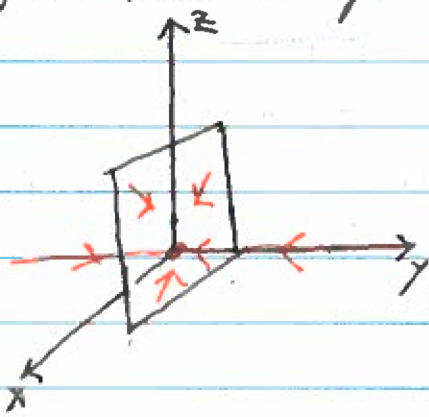
$$\dot{V} = \frac{1}{\sigma}x\dot{x} + y\dot{y} + z\cdot\dot{z}$$
$$= xy - x^2 + rxy - y^2 - xyz + xyz - bz^2$$
$$= -(x^2 - (r+1)xy - y^2) - bz^2$$

$$\Rightarrow \dot{V} = -\left(x^2 - (r+1)xy + \frac{(r+1)^2}{4}y^2\right) - \left(1 - \frac{(r+1)^2}{4}\right)y^2 - bz^2$$
$$= -\left(x - \frac{(r+1)}{2}y\right)^2 - \frac{1}{4}(3 - 2r - r^2)y^2 - bz^2$$
$$= -\left(x - \frac{(r+1)}{2}y\right)^2 - \frac{1}{4}(3+r)(1-r)y^2 - bz^2$$

Therefore, if $r<1$ V is a Lyapunov function.

1. $\dot{V} \leq 0$, with $\dot{V} = 0$ only if $x, y, z = 0$
2. $V \geq 0$
3. $\lim_{t\to\infty} V(t) = 0 \Rightarrow \lim_{t\to\infty} (x(t), y(t), z(t)) = (0,0,0)$.

If $r<1$, all tractories go to the origin



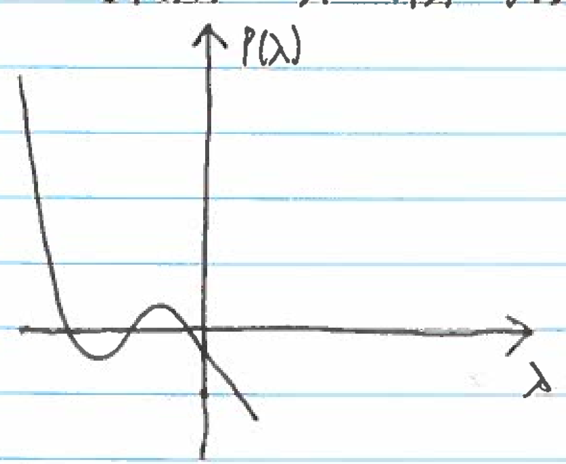
Case 2 $(1<r<r_s)$

$(0,0,0)$ has eigenvalues $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$.

$\rightarrow$ This is like a saddle node.

We denote the other two fixed points by C_+, C_-. The characteristic polynomial is

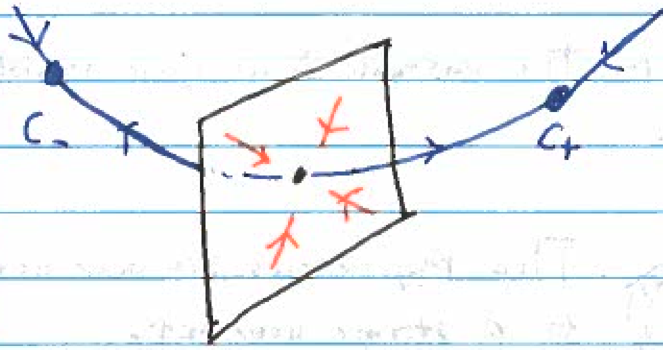
$$3P(\lambda) = -\lambda^3 - 41\lambda^2 - 88\lambda - 80(r-1)$$



$\Rightarrow$ Changing r shifts the graph.

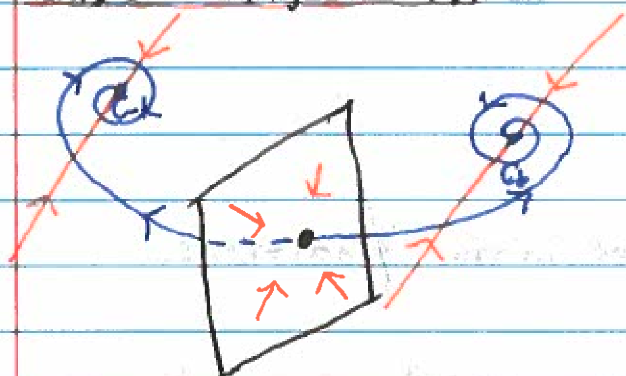
$\Rightarrow$ There exists r_s such that

$1 < r < r_s \Rightarrow C_\pm$ are stable.



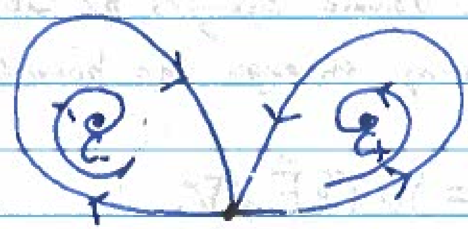
$\Rightarrow$ System has two steady state convection rolls.

Case 3 $(r_s < r < r_0)$:



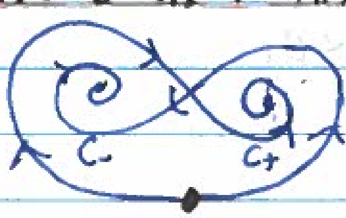
Stability analysis tells us $C_\pm$ are stable spirals around the fixed points.

Case 4 $(r = r_0)$:



Unstable limit cycle born in a Homoclinic bifurcation.

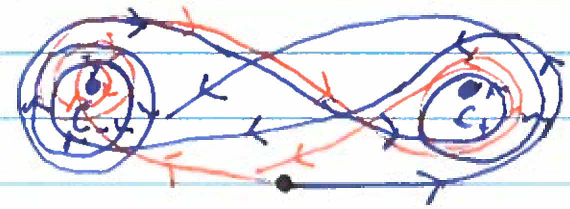
Case 5 $(r_0 < r < r_H)$:



The spirals loop around both fixed points. Impossible to predict which fixed point trajectories go to! This creates a strange invariant set.

Case 6 ($r > r_H$):

A Hopf bifurcation occurs at r_H. The unstable limit cycle vanishes leaving behind two unstable spirals.



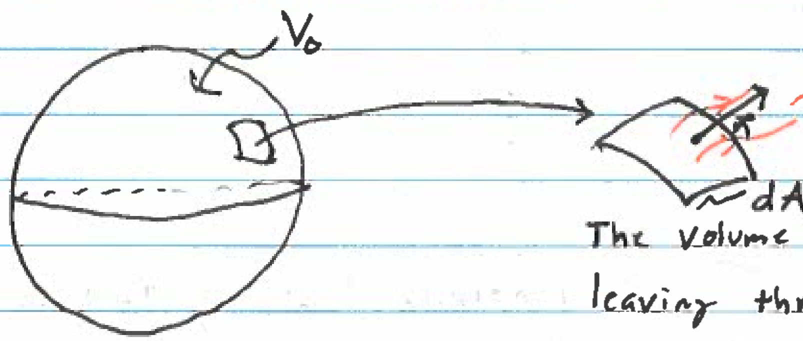
The trajectories are now attracted to a strange new set.

Volume Contraction:

Consider

$$\dot{\vec{x}} = F(x),$$

let V_0 denote an initial volume of initial conditions.



The volume changes by what is leaving through the boundary.

Liouville's Theorem - $\dot{V} = \int_{\partial V} \vec{F} \cdot \vec{n} \, dA = \int_V \nabla \cdot \vec{F} \, dV.$

Consequence for Lorenz equations:

$$\vec{F} = \begin{bmatrix} \sigma(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix} \Rightarrow \nabla \cdot F = -\sigma - 1 - b < 0.$$

$$\Rightarrow \dot{V} = \int_V (-\sigma - 1 - b) \, dV = -(\sigma + 1 + b) V$$

$$\Rightarrow V = V_0 e^{-(\sigma+1+b)t}.$$

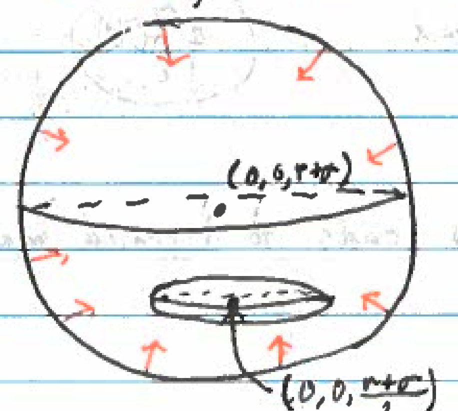
All volumes shrink to nothing! However, trajectories must be attracted to something.

The attractor
1. Can't be a fixed point if $r > r_H$
2. Can't be a closed surface, which eliminates quasi-periodicity
3. Possible to eliminate limit cycles.

Goals:
1. Find out what new objects is.
2. Try to define and study chaos.

Trajectories Remain Bounded:

Consider a sperical urface $S_R = x^2+y^2+(z-r-\sigma)^2 = R^2$.



For a trajectory starting on the surface we have

$$\frac{d}{dt}[R^2] = \frac{d}{dt}[x^2+y^2+(z-r-\sigma)^2] = 2x\dot{x}+2y\dot{y}+2(z-r-\sigma)\dot{z}.$$

Since $\dot{x} = \sigma(y-x)$, $\dot{y} = rx-y-xz$, $\dot{z} = xy-bz$ we have that

$$\frac{d}{dt}R^2 = 2x\sigma(y-x)+2y(rx-y-xz)+2(z-r-\sigma)(xy-bz)$$
$$= -2\left[\sigma x^2+y^2+b\left(z-\frac{r+\sigma}{2}\right)^2-\frac{b(r+\sigma)^2}{4}\right]$$

Pick R enough so that the ellipse

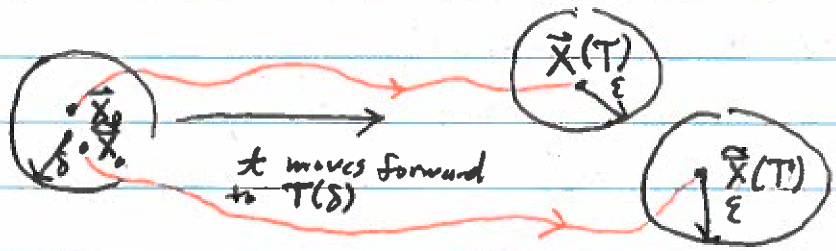
$$rx^2+y^2+b\left(z-\frac{r+\sigma}{2}\right)^2 = \frac{b(r+\sigma)^2}{4}$$

is enclosed in S_R. This guarantees S_R is a trapping region.

What is Chaos?

Chaos: Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic - There exist trajectories that do not go to fixed points, periodic orbits, or quasi-periodic orbits.
2. Sensitive Dependence - We say $\dot{\vec{x}} = F(\vec{x})$ has sensitive dependence on initial data on a set A if $\forall \vec{x}_0 \in A$, $\exists \varepsilon(\vec{x}_0)$ such that $\forall \delta > 0$, $\exists (\hat{\vec{x}}_0(\delta)$ and $T(\delta) > 0)$ with $\|\vec{x}_0(\delta) - \hat{\vec{x}}_0(\delta)\| < \delta$ and $\|\vec{x}(T) - \hat{\vec{x}}(T)\| > \varepsilon$.



Trying to show both these quantities leads to iterated maps and fractals.