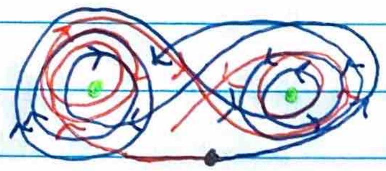
Lecture #3: Lorenz Map and Iterated Maps

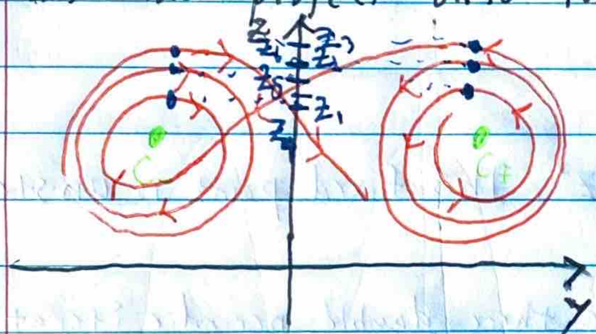
Recall the Lorenz equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

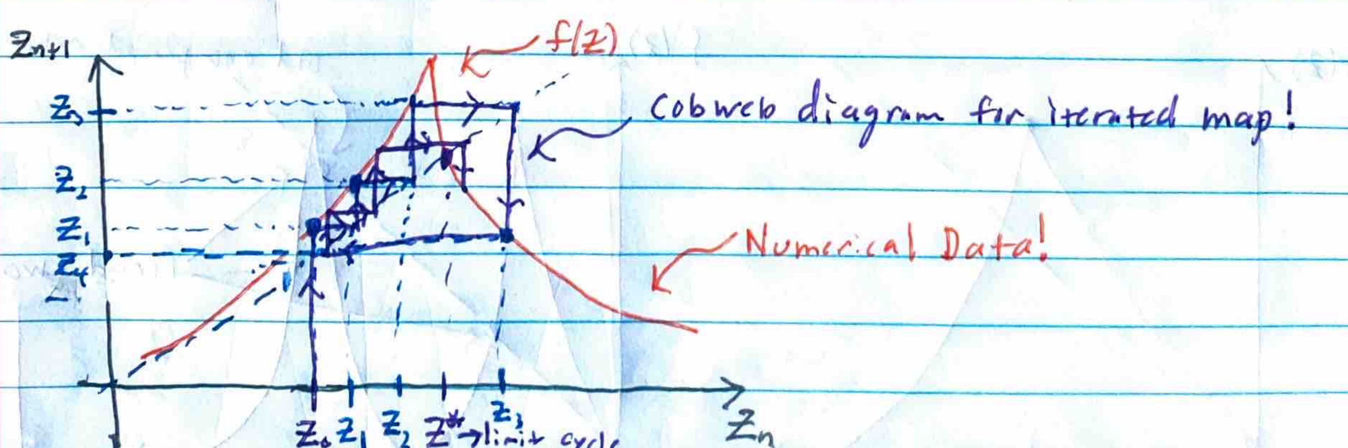
If $r > r_H$, the generic dynamics looks like



If we project onto the y-z plane we have:



Lorenz's idea was to keep track of each local maximum and see if one could predict their relationship.

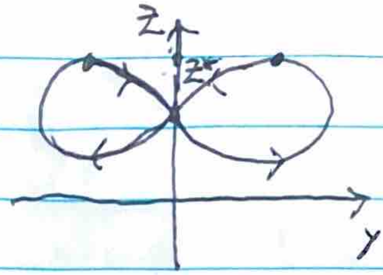


This observation is amazing! We can extract some order from chaos by studying the iterated map

$$z_{n+1} = f(z_n)$$

Note, by construction $|f'(z)| > 1$.

There is a closed orbit at $z \doteq z^*$ which corresponds to a limit cycle.



Now, consider $z_0 = z^* + \delta_0$, δ_0 = initial perturbation.

$$\Rightarrow z_1 = f(z^* + \delta_0)$$
$$\approx f(z^*) + f'(z^*)\delta_0$$
$$= z^* + f'(z^*)\delta_0$$
$$\Rightarrow |z_1 - z^*| = |f'(z^*)||\delta_0| > |\delta_0|$$

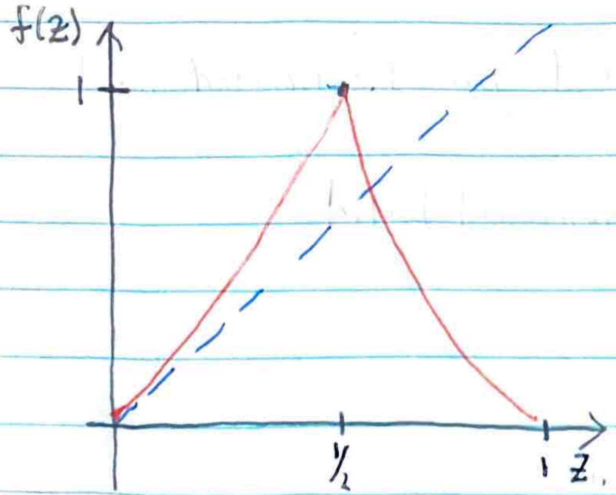
If we let $\delta_1 = z_1 - z^*$ then

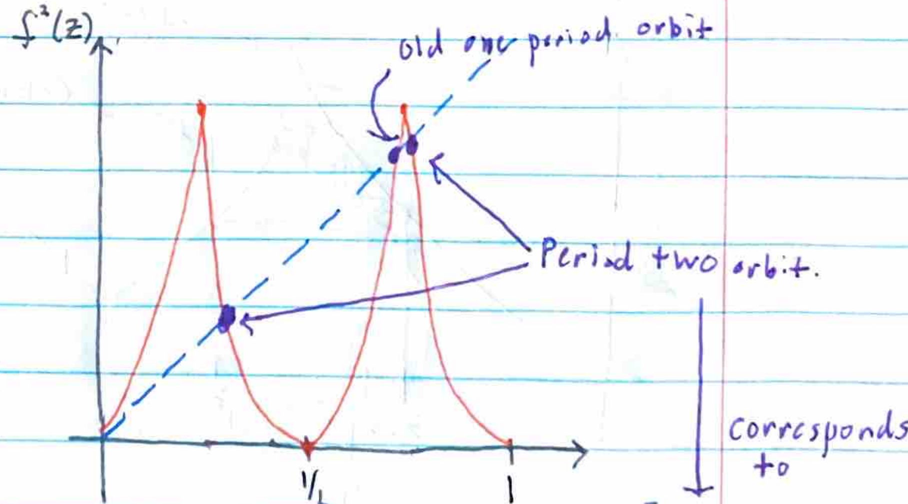
$$|\delta_1| > |\delta_0|,$$

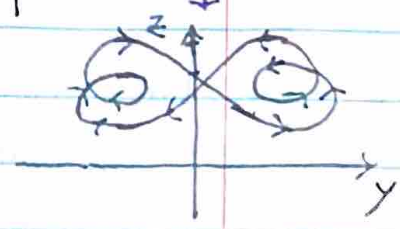
i.e., the trajectory moves away from z^*. This fixed point is unstable and hence the limit cycle is unstable.

* What about other limit cycles? Are there double periodic iterations?

$$z_{n+2} = f(f(z_n)) = f^2(z).$$





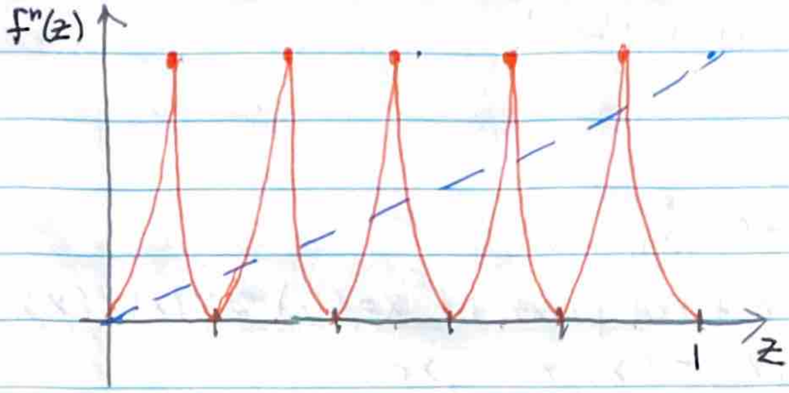


Note:

$$\frac{d}{dz} f^2(z) = \frac{d}{dz} f(f(z)) = f'(f(z))f'(z)$$

$$\Rightarrow \left|\frac{d}{dz} f^2(z)\right| = \left|f'(f(z))f'(z)\right| > 1 \cdot 1 = 1. \Rightarrow$$ The period two orbit is unstable.

In the General Case:



There exist a countably infinite number of unstable limit cycles!

What is an Attractor?

An attractor is a closed set A with the following properties.

1. A is invariant: Any $\vec{x}(t)$ that starts in A stays in A for all time.
2. A attracts an open set of initial conditions: There exists an open set U containing A such that if $\vec{x}(0) \in A$, then $\|\vec{x}(t) - A\| \to 0$ as $t \to \infty$. The largest U is called the basin of attraction.
3. A is minimal: There is no proper subset of A satisfying 1-2

Holds if time is discrect as well

Attractor for Lorenz Map?

If we let $\{\bar{z}_1^1, \bar{z}_1^2, \bar{z}_2^2, \bar{z}_1^3, \bar{z}_2^3, \bar{z}_3, \ldots\} = Z^*$ denote the fixed points of $f, f^2, f^3, \ldots$ the the attractor A is the uncountable set

$$A = (Z^*)^c.$$

Discrete Dynamical Systems

$f: \mathbb{R} \mapsto \mathbb{R}$, $f: [0,1] \mapsto [0,1]$, $f: S^1 \mapsto S^1$

$$x_{n+1} = f(x_n)$$

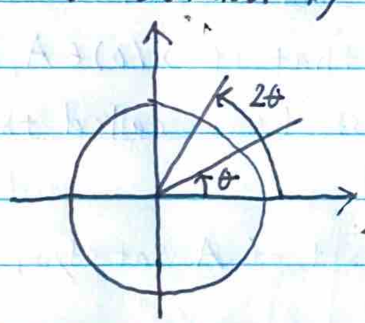
1. Orbit of x_0: $\gamma(x_0) = \{x_0, x_1, \dots\}$
2. Fixed points: x is a fixed point if $x = f(x) \rightarrow \gamma(x) = \{x\}$.
3. Period k orbits: $\gamma(x_0) = \{x_0, x_1, \dots, x_k\}$.

Theorem – Let x^* be a fixed point of f and assume f is differentiable near x^*. Then

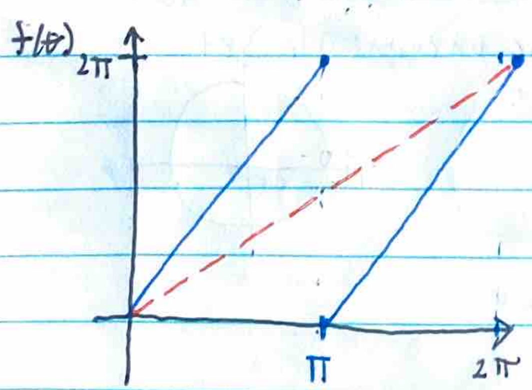
$$x^* \text{ is } \begin{cases} \text{stable if } |f'(x^*)| < 1 \\ \text{unstable if } |f'(x^*)| > 1 \end{cases}$$

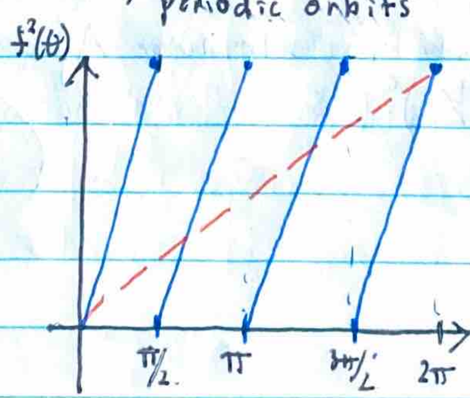
Example:

$f: S^1 \mapsto S^1$ defined by $\theta \mapsto 2\theta$.



1. Since $\frac{d}{d\theta}(f^n(\theta)) = \frac{d}{d\theta}(2^n\theta) = 2^n$, all periodic orbits are unstable.
2. Let $P_k = \{\theta \in S^1 : f^k(\theta) = \theta\}$, i.e., set of not necessarily smallest periodic orbits





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3. Let $P = \bigcup_{k=1}^{\infty} P_k$, i.e. all periodic orbits. Now, $\theta \in P_k$ if $2^k\theta = \theta + 2\pi n$ for some $n \in \mathbb{N}$, with $0 \le n \le 2^{k-1}$.

$$\Rightarrow \theta = \frac{2\pi n}{2^k - 1}$$

In particular:

$P_1 = \{2\pi\}$ → 1 element

$P_2 = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}\right\}$ → 3 elements

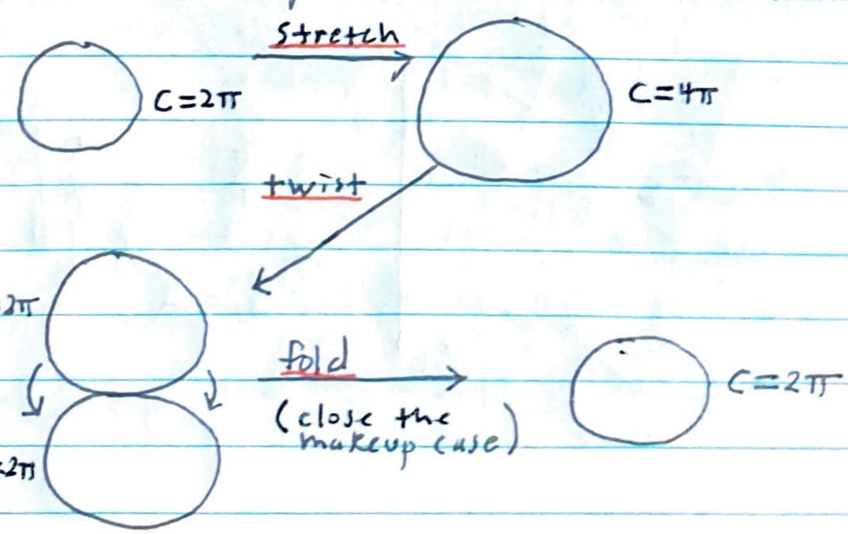
$P_3 = \left\{\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}\right\}$ → 7 elements

⋮

4. P is dense in S meaning for all $\varphi \in S$ and for all $\varepsilon > 0$ there exists $\theta \in P$ such that $|\theta - p| < \varepsilon$.

Definition (Topologically Transitive): The attractor is indecomposable meaning for any two open sets $O_1, O_2 \subset S^1$, there exists $n > 0$ with $f^n(O_1) \cap O_2 \neq \emptyset$.

The circle map can be viewed as follows



Definition: A discrete dynamical system

$$x_{n+1} = f(x_n)$$

is chaotic on an attractor A if

- It has sensitive dependence on initial conditions.
- The set of periodic orbits is dense on A.
- A is topologically transitive.