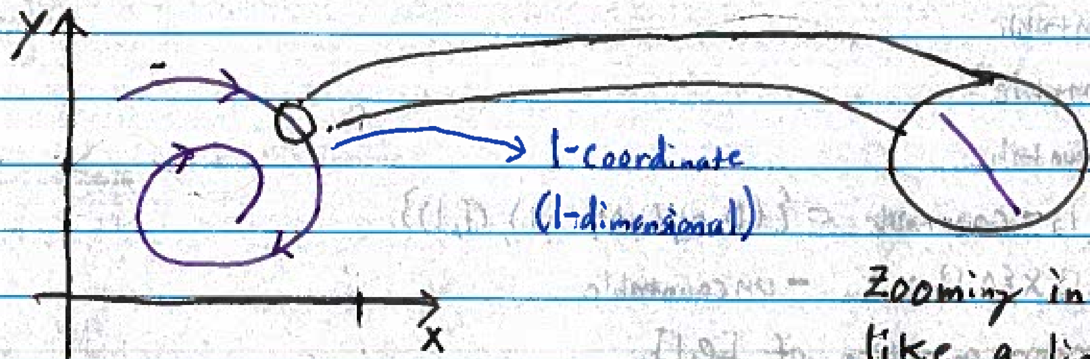
Lecture #4: Fractals

Dimension - How do we measure dimension of a set? One idea is to count the number of coordinates need to describe a set.

Example:

Let $\vec{f}:[0,1]\to\mathbb{R}^2$ be smooth and $S=\{(x,y)\in\mathbb{R}^2 : (x,y)=\vec{f}(t)\}$

Then S is one-dimensional. A natural coordinate is the arclength along the curve $L(t)=\int_0^t \|f\|\,ds$.



Zooming in looks like a line.

Example:

$$f(x)=\sum_{k=1}^{\infty}\frac{\sin(\pi k^3 x)}{\pi k^2}, \text{ on the interval } [0,1].$$

$$|f(x)|\le\sum_{k=1}^{\infty}\frac{1}{\pi k^2}=\frac{\pi}{6k^2}.$$

By the Weirstrass-M test $f(x)$ is well defined and continuous. However,

$$f'(x) \text{ "=" } \sum_{k=1}^{\infty}\cos(\pi k^3 x)$$

For large k, $\cos(\pi k^3 x)\approx\pm1$ infinitely often. Therefore, we expect

$$|f'(x)|=\infty$$

and thus f is not differentiable almost everywhere.

Consequence: The graph of f has infinite arclength.

Consequence: The graph of f is not 1-dimensional. Dimension for this object cannot be defined in the classical sense.

Size of Sets

Countable: Finite or can be put into 1-1 correspence with $\mathbb{N}$.

Uncountable: Not countable.

Examples:

$\mathbb{Z}$ - countable

$\mathbb{Q}$ - countable

$\mathbb{Z} \times \mathbb{Z}$ - countabe

$\{0,1\} \times \{0,1\}$ - countable $= \{(0,0),(1,0),(0,1),(1,1)\}$.

$\{0,1\} \times \{0,1\} \times \{0,1\} \times \dots$ - uncountable

$\downarrow$

Binary representation of $[0,1]$.

Set of Measure 0: A set S has measure 0 in $\mathbb{R}^n$ if $\forall \varepsilon > 0$, S is a subset of a union of open cubes the sum of whose volume is less tham ε.

Example:

$\mathbb{Q}$ is a set of measure 0 in $\mathbb{R}$.

~~proof~~:

We can index $\mathbb{Q}$ by points $\{r_1, r_2, \dots\}$. Let $b_i = (r_i - \frac{\varepsilon}{2^i}, r_i + \varepsilon/2^i)$.

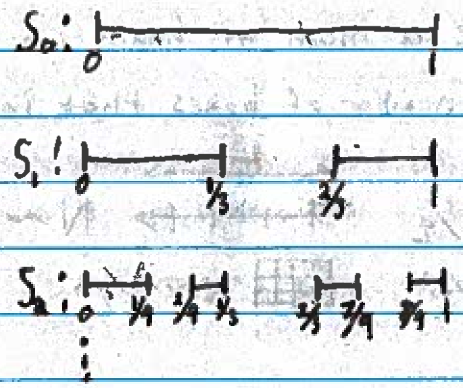
Then

$$V\left(\bigcup_{i=1}^{\infty} b_i\right) \leq \sum_{i=1}^{\infty} V(b_i) = \sum_{i=1}^{\infty} 2\varepsilon/2^i = C \cdot \varepsilon$$

Consequence: There are two ways to measure the size of a set.

Example: Cantor Set

The cantor set is formed by removing middle third of each set



$$S_\infty = \bigcap_{n=1}^{\infty} S_n = \text{Cantor Set}$$

1. S_∞ is uncountable: If $x \in S_\infty$, then $x \in S_1, S_2, \ldots$. We can assign a sequence of + and − to x by whether x moves to the left or rightmost interval at each stage. That is for the sequence

   $(+, -, \ldots)$

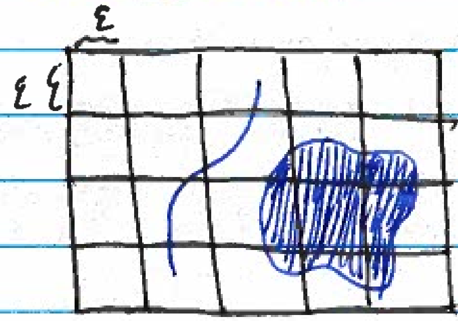
   this would mean

   $x \in [2/3, 1]$ and $x \in [2/3, 5/9]$

   and so on. Consequently, S_∞ is in 1-1 coorespondence with sequences of the form $(+, -, +, \ldots)$ which can be put into 1-1 coorespondence with binary.

2. S_∞ has measure 0: Cover by balls of volume $\left(\frac{1}{3}\right)^n 2^n = \left(\frac{2}{3}\right)^n$. take limit $n \to \infty$.

Box Dimension:



Let $A \subset \mathbb{R}^n$. Take a mesh of boxes of length ε.
Let $N(\varepsilon)$ be the number of boxes that intersect with A.

1-dim: $N(\varepsilon) \sim 1/\varepsilon$ $\qquad$ L, ε $\qquad$ $N \sim L/\varepsilon$

2-dim: $N(\varepsilon) \sim 1/\varepsilon^2$ $\qquad$ $N \sim A/\varepsilon^2$

Assuming there is some scaling law

$$N(\varepsilon) \sim 1/\varepsilon^{d(\varepsilon)}$$

then

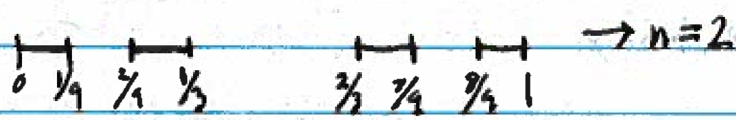
$$\ln(N(\varepsilon)) = d(\varepsilon)\ln(1/\varepsilon) + C$$

$$\Rightarrow d(\varepsilon) = \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)} - \frac{C}{\ln(1/\varepsilon)}$$

$$\Rightarrow \boxed{d = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)} \quad \text{box-dimension}}$$

Example:

For the Cantor set we can construct a sequence of coverings. Let $\varepsilon_n = (1/3)^n$ be width of boxes.



the $N(\varepsilon) = (2)^n$

$$d = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln(2)}{\ln(3)}$$

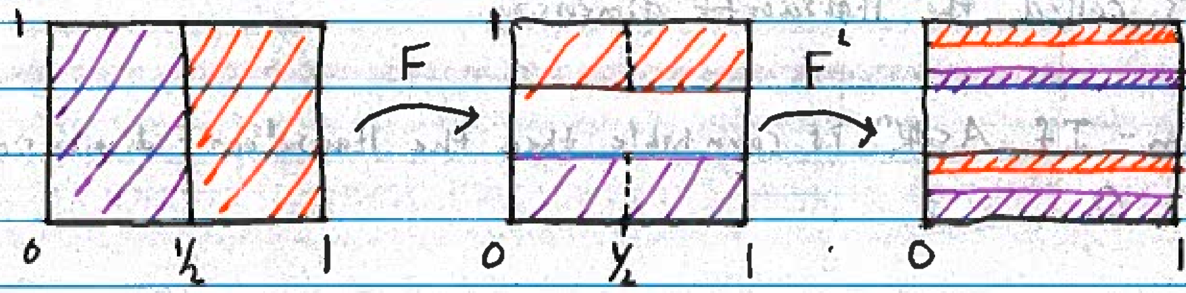
Example:

If we let $S = \mathbb{Q} \cap [0,1]$, no matter how you cover the set $N(\varepsilon) = 1/\varepsilon$

$$\Rightarrow d = 1.$$

Example:

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, \ y_n/3), & 0 \le x_n \le 1/2 \\ (2x_n - 1, \ y_n/3 + 2/3), & 1/2 \le x_n \le 1 \end{cases}$$



The attracting set is

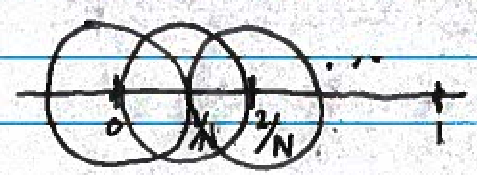
$A = [0,1] \times C \to C$ is the Cantor set.

Box dimension is $d = 1 + \dfrac{\ln(2)}{\ln(3)}$

Hausdorff Dimension:

Intuition:

Cover a set with disks



$A \sim \pi r^2 N$ (2) → 2-dimensional measure. $, \ r \sim 1/N$

$A \sim \dfrac{\pi}{N} \to 0$ as $N \to \infty$. Howerewer, what if we changed power?

$H \sim \pi r^1 N$

$\sim \pi \to 1$ dimensional object.

Let $\Gamma(\varepsilon)$ be set of coverings of $A \subset \mathbb{R}^n$ of closed balls B_i of radii $r_i \le \varepsilon$. Set:

- $H_\alpha(A, \varepsilon) = \inf_{\gamma \in \Gamma(\varepsilon)} \sum r_i^\alpha$
- $H_\alpha = \lim_{\varepsilon \to 0} H_\alpha(A, \varepsilon)$ (Hausdorff measure)

There exists unique $d_H \geq 0$ such that

$$H_\alpha(A) = \begin{cases} 0, & \alpha > d_H \\ \infty, & \alpha < d_H \end{cases}$$

d_H is called the Hausdorff dimension.

Theorem - If $A \subset \mathbb{R}^n$ is countable then the Hausdorff dimension of A is 0.

proof:

Since A is countable we can construct a sequence $x_i \in A$ such that $A = \bigcup_{i \geq 1}^{\infty} x_i$. For $\varepsilon > 0$, let $\gamma(\varepsilon)$ be a covering of A by closed balls centered at x_i of radii $\varepsilon/2^i$. Then,

$$H_\alpha(A, \varepsilon) \leq \sum_{i=1}^{\infty} \frac{\varepsilon^\alpha}{(2^i)^\alpha} = \sum_{i \geq 1}^{\infty} \frac{\varepsilon^\alpha}{(2^\alpha)^i} = C \cdot \varepsilon^\alpha.$$

Consequently,

$$H_\alpha(A) = \lim_{\varepsilon \to 0} H_\alpha(A, \varepsilon) = 0$$

for all $\alpha > 0$. Therefore, $d_H = 0$.

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