Lecture #8: Weak Van der Pol Oscillator

$$\ddot{x} + x = \varepsilon(1-x^2)\dot{x}$$

Phase Portrait Analysis

$$\dot{x} = v$$
$$\dot{v} = -x + \varepsilon(1-x^2)v$$

Nullclines:

$v = 0 \quad (\dot{x} = 0)$

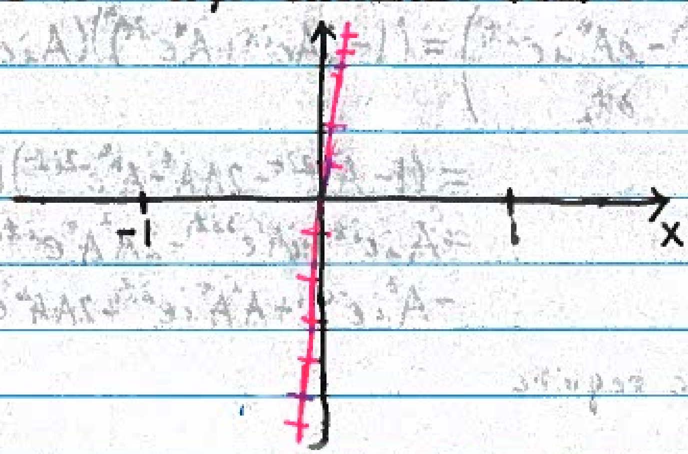
$v = \dfrac{x}{\varepsilon(1-x^2)} \quad (\dot{v} = 0)$

Jacobian:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$$

$$\lambda_{1,2} = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$

Thus, for $\varepsilon \ll 1$ the fixed point is an unstable spiral. Moreover, there is a Hopf bifurcation when $\varepsilon = 0$.



The phase portrait is too hard to draw by hand for $\varepsilon \ll 1$.

We can use method of multiple scales to determine if the Hopf bifurcation is sub or supercritical.

Method of Multiple Scales

$$\ddot{x} + x = \varepsilon(1-x^2)\dot{x}$$

Assume two time scales

$$t_1 = t, \quad t_2 = \varepsilon t.$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_1^2} + 2\varepsilon\frac{\partial^2}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2}{\partial t_2^2}$$

To $O(\varepsilon)$ we obtain the PDE:

$$\frac{\partial^2 x}{\partial t_1^2} + 2\varepsilon\frac{\partial^2 x}{\partial t_1 \partial t_2} + x = \varepsilon(1-x^2)\frac{\partial x}{\partial t_1}$$

We assume

$$x = x_0 + \varepsilon x_1 + \dots$$

Substituting into the PDE we obtain at $O(1)$:

$$\frac{\partial^2 x_0}{\partial t_1^2} + x_0 = 0$$

$$\Rightarrow x_0 = A(t_2)e^{it_1} + A^*(t_2)e^{-it_1}$$

At $O(\varepsilon)$ we have:

$$\frac{\partial^2 x_1}{\partial t_1^2} + x_1 + 2\left(\frac{\partial A}{\partial t_2} ie^{it_1} - \frac{\partial A^*}{\partial t_2} ie^{-it_1}\right) = (1-(Ae^{it}+A^*e^{-it})^2)(Aie^{it_1} - A^*ie^{-it_1})$$

$$= (1 - A^2e^{2it} - 2AA^* - A^{*2}e^{-2it})(Aie^{it_1} - A^*ie^{-it_1})$$

$$= Aie^{it_1} - iA^3e^{3it_1} - 2A^2A^*e^{it_1} - iA^{*2}Ae^{-it_1}$$
$$- A^*ie^{-it_1} + A^2A^*ie^{it_1} + 2AA^{*2}e^{-it_1} + A^{*3}e^{-3it_1}$$

To remove secular terms we require

$$2i\frac{\partial A}{\partial t_2} = Ai - A^2A^*i$$

$$\Rightarrow \frac{\partial A}{\partial t_2} = \frac{1}{2} A - A^2 A^*.$$

We express A in exponential form to yield:

$$A = re^{i\theta}$$

$$\Rightarrow \frac{\partial A}{\partial t_2} = \frac{\partial r}{\partial t_2} e^{i\theta} + ir\frac{\partial \theta}{\partial t_2} e^{i\theta} = \frac{1}{2}(re^{i\theta} - r^3 e^{i\theta})$$

$$\Rightarrow \frac{\partial r}{\partial t_2} = \frac{1}{2}(r - r^3), \quad \frac{\partial \theta}{\partial t_2} = 0$$

In the long run, $r \to 1$. and $\theta = \theta_0$. Consequently, the limit cycle is approximated by:

$$x = e^{i\theta_0} e^{it} + e^{-i\theta_0} e^{it}$$

$$= 2\cos(\theta_0 + t).$$

The radius is 2 and the period is 2π.