# Lecture #9: Boundary Layers

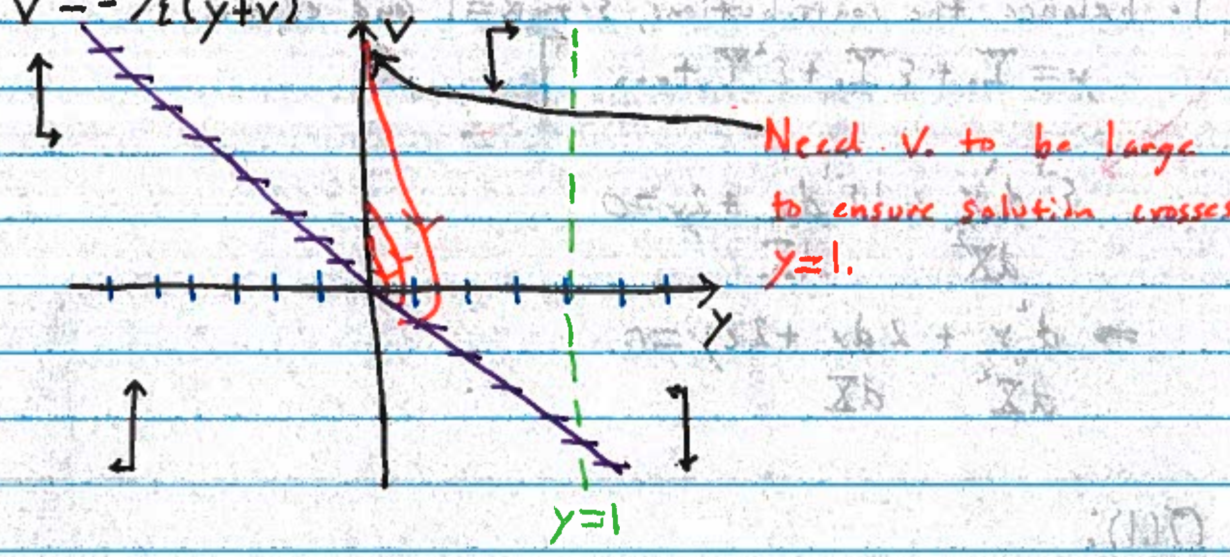
## Example:

$$\varepsilon y'' + 2y' + 2y = 0$$ , ' means differentiation with respect to x.

$$y(0) = 0, \quad y(1) = 1$$

Lets draw the phase portrait to get an idea of system behavior

$$y' = v$$
$$v' = -\frac{2}{\varepsilon}(y+v)$$




Intially, v must be very large. Also,

$$y'' = \frac{dv}{dx} = \frac{-2}{\varepsilon}(y+v)$$

Which is also large.

## Naive Expansion (Outer Expansion)

Assume $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$

$$\Rightarrow \varepsilon(y_0'' + \varepsilon y_1'' + \ldots) + 2(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \ldots) + 2(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots) = 0$$

### O(1):

$$y_0' + y_0 = 0 \Rightarrow y_0 = ae^{-x}$$

This cannot satisfy both boundary conditions. If we satisfy the condition at x=1, we have $a = e \Rightarrow y_0(x) = e^{1-x}$.

## Inner Expansion

Rescale near $x=0$ to resolve the large derivatives in $y$. So, we rescale by $X = \varepsilon^{-\alpha} x$:

$$\frac{d}{dx} = \frac{\partial X}{\partial x}\frac{d}{dX} = \varepsilon^{-\alpha}\frac{d}{dX}, \quad \frac{d^2}{dx^2} = \varepsilon^{-2\alpha}\frac{d^2}{dX^2}$$

$$\Rightarrow \varepsilon^{1-2\alpha}\frac{d^2y}{dX^2} + 2\varepsilon^{-\alpha}\frac{dy}{dX} + 2y = 0$$

To balance the contributions, set $\alpha = 1$ and expand:

$$y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \dots$$

$$\varepsilon^{-1}\frac{d^2y}{dX^2} + 2\varepsilon^{-1}\frac{dy}{dX} + 2y = 0$$

$$\Rightarrow \frac{d^2y}{dX^2} + 2\frac{dy}{dX} + 2\varepsilon y = 0$$

$O(1)$:

$$\frac{d^2Y_0}{dX^2} + 2\frac{dY_0}{dX} = 0$$

$$Y_0(0) = 0$$

$$\Rightarrow Y_0(X) = A(1-e^{-2X}) = A(1-e^{-2x/\varepsilon})$$

## Matching

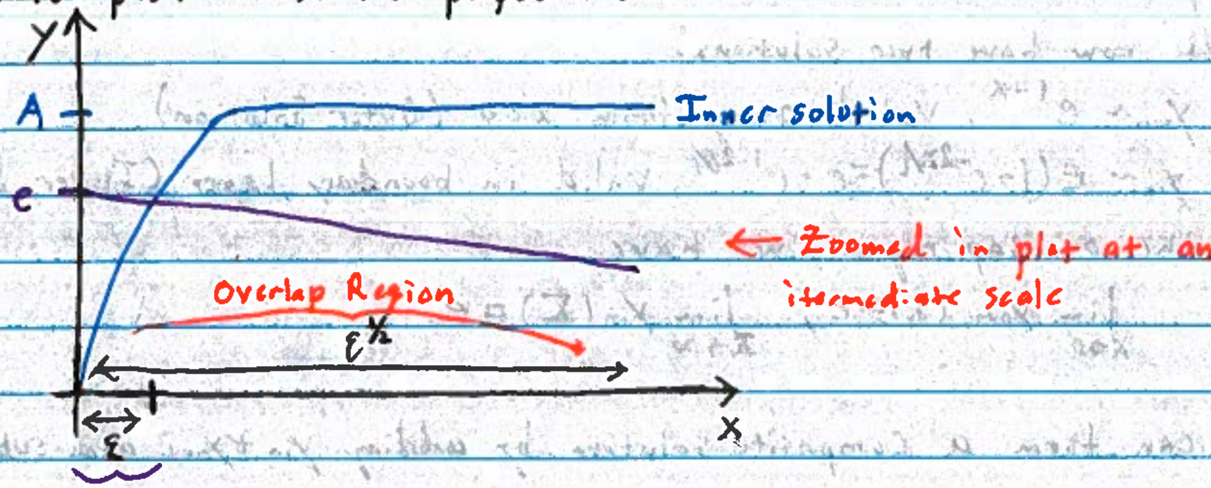
We have two proposed solutions which both don't match boundary conditions:

$$y \sim y_0(x) = e^{1-x} \quad \text{(Outer Solution)}$$

$$y \sim Y_0(X) = Y_0(x/\varepsilon) = A(1-e^{-2x/\varepsilon}) \quad \text{(Inner Solution)}$$

* The key observation is that both expansions approximate the same function and thus must agree in an overlap region

Lets plot these two proposed solutions




Boundary layer: A small region where the inner solution is valid.

To match on the intermediate scale we set $A = e$. How do we make this more precise?

Introduce an intermediate variable $\eta = x/\sqrt{\varepsilon}$ and require that that the solutions coincide as $\varepsilon \to 0$ in the overlap region:

$$\lim_{\varepsilon \to 0^+} y_0(\sqrt{\varepsilon}\,\eta) = \lim_{\varepsilon \to 0^+} Y_0(\sqrt{\varepsilon}\,x)$$

$$\Rightarrow \lim_{\varepsilon \to 0^+} e^{1-\varepsilon^{1/2}\eta} = \lim_{\varepsilon \to 0^+} A\left(1 - e^{-2\eta/\varepsilon^{1/2}}\right)$$

$$\Rightarrow e = A.$$

Note, we could have equivalently written these limits as

$$\boxed{\lim_{x \to 0} y_0(x) = \lim_{X \to \infty} Y_0(X)}$$

## Composite Solution

We now have two solutions:

$y_{out} \sim e^{1-x}$, valid away from $x=0$ (Outer Solution)

$y_{in} \sim e(1-e^{-2X/\varepsilon}) = e - e^{1-2X/\varepsilon}$, valid in boundary layer (Inner Solution)

In the overlap region, we have

$$\lim_{x\to 0} y_{out}(x) = e, \quad \lim_{X\to\infty} y_{in}(X) = e.$$

We can form a composite solution by adding $y_{in} + y_{out}$ and subtracting the common term

$$y \sim e^{1-x} + e(1-e^{-2X/\varepsilon}) - e$$

$$\Rightarrow \boxed{y_{comp} \sim e^{1-x} - e^{1-2X/\varepsilon}}$$

Note,

$$y_{comp}(0) = 0 \quad \text{but} \quad y_{comp}(1) = 1 - e^{-2/\varepsilon}$$

and thus the boundary conditions are not exactly satisfied.

However, for all $p \in \mathbb{R}$:

$$\lim_{\varepsilon\to 0} \frac{1 - y_{comp}(1)}{\varepsilon^p} = \lim_{\varepsilon\to 0} \frac{1-e+e^{-2/\varepsilon}}{\varepsilon^p} = \lim_{z\to\infty} \frac{1-1+e^{-2z}}{(1/z^p)} = \lim_{z\to\infty} \frac{z^p}{e^{2z}} = \lim_{z\to\infty} \frac{p!}{2^p e^{2z}} = 0$$

and thus $y_{comp}(1)$ is transcendentally close to solving the boundary conditions.

Example:

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^x$$

$$y(0)=2, \quad y(1)=1$$

Outer Solution

Try an expansion of the form

$$y_{out} = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$$

$O(1)$:

$$y_0 = e^x$$

Apparently, we cannot satisfy either boundary condition. We need two boundary layers.

Left Layer

$$X = \varepsilon^{-\alpha} x$$

$$\Rightarrow \varepsilon^{2-2\alpha} \frac{d^2 y}{dX^2} + \varepsilon X \frac{dy}{dX} - y = -e^{\varepsilon^\alpha X}$$

Balancing we have that

$$2 - 2\alpha = 0$$

$$\Rightarrow \alpha = 1 \quad \text{(Boundary Layer of Width } \varepsilon\text{)}$$

$$\Rightarrow \frac{d^2 y}{dX^2} + \varepsilon X \frac{dy}{dX} - y = -1$$

Assume:

$$y_L = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \dots$$

$O(1)$:

$$\frac{d^2 Y_0}{dX^2} - Y_0 = -1, \quad Y_0(0) = 2$$

$$\Rightarrow y_0 = 1 + Ae^{-X} + (1-A)e^{X}$$

$$\Rightarrow y_{left} \sim 1 + Ae^{-x/\varepsilon} + (1-A)e^{x/\varepsilon}$$

To match, we clearly need $A=1$ or else

$$\lim_{X\to\infty} y_{left}(X) = \infty.$$

Right Layer

$$\tilde{x} = \frac{x-1}{\varepsilon^{\beta}}$$

$$\Rightarrow \varepsilon^{2-2\beta}\frac{d^2y}{d\tilde{x}^2} + (1+\varepsilon^{\beta}\tilde{x})\varepsilon^{1-\beta}\frac{dy}{d\tilde{x}} - y = -e^{1+\varepsilon^{\beta}\tilde{x}}, \quad y(\tilde{x}=0)=1$$

Balancing $2-2\beta = 1-\beta \Rightarrow \beta = 1.$

$$\Rightarrow \frac{d^2y}{d\tilde{x}^2} + (1+\varepsilon)\frac{dy}{d\tilde{x}} - y = -e^{(1+\varepsilon\tilde{x})}$$

Assume

$$y_R = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots$$

$O(1)$:

$$\frac{d^2y_0}{d\tilde{x}^2} + \frac{dy_0}{d\tilde{x}} - y_0 = -e, \quad y_0(0)=1$$

$$y_0(\tilde{x}) = e + B\exp\left(\frac{-1+\sqrt{5}}{2}\tilde{x}\right) + (1-e-B)\exp\left(\frac{-1-\sqrt{5}}{2}\tilde{x}\right)$$

To match with the outer solution, we again need $1-e-B=0 \Rightarrow B = 1-e.$

## Composite Solution

We have three proposed solutions

$$y_{outer} = e^{x}$$

$$y_{left} = 1 + e^{-X} = 1 + e^{-x/\varepsilon}$$

$$y_{right} = e + (1-e)\exp\left(\frac{-1+\sqrt{5}}{2}\tilde{X}\right) = e + (1+e)\exp\left(\frac{-1+\sqrt{5}}{2}\left(\frac{x-1}{\varepsilon}\right)\right)$$

The overlapping values of the functions are given by:

$$\lim_{x\to 1} y_{outer}(x) = \lim_{\tilde{X}\to -\infty} y_{right}(\tilde{X}) = e$$

$$\lim_{x\to 0} y_{outer}(x) = \lim_{X\to \infty} y_{left}(X) = 1$$

Therefore, the composite solution is

$$y \sim y_{outer} + y_{left} + y_{right} - e - 1$$

$$\boxed{y \sim e^{x} + e^{-x/\varepsilon} + (1-e)\cdot\exp\left(\frac{-1+\sqrt{5}}{2}\left(\frac{x-1}{\varepsilon}\right)\right)}$$

## Example:

$$\varepsilon y'' = yy' - y$$

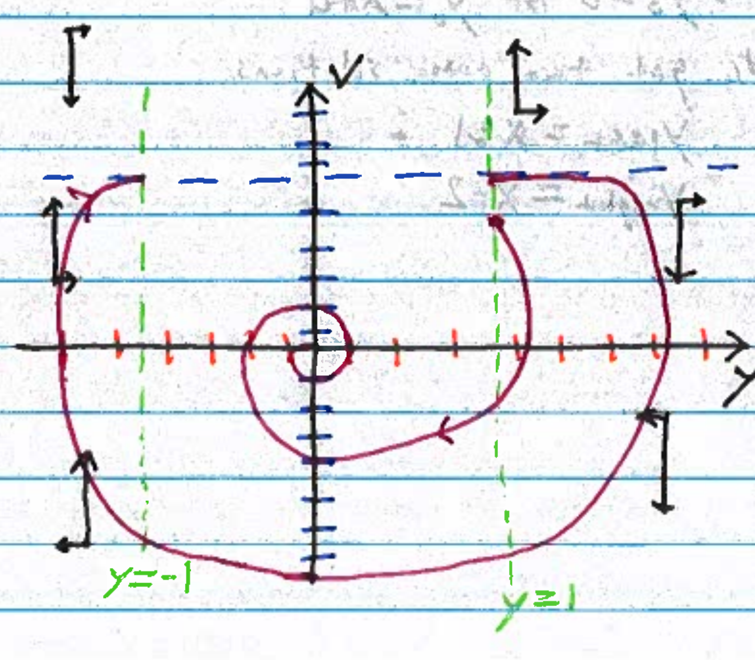
$$y(0) = 1, \quad y(1) = -1$$

phase portrait

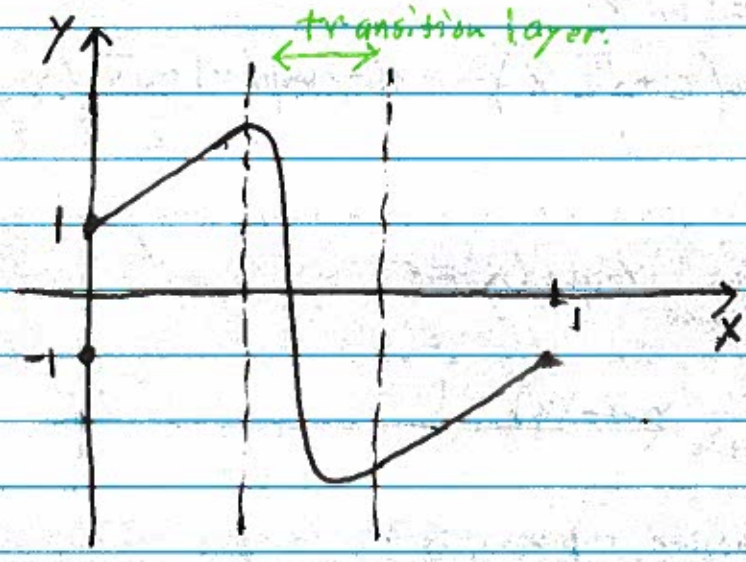
$$y' = v$$

$$v' = \tfrac{1}{\varepsilon}\, y(v-1)$$

$v = 1$ is slow manifold



On the slow manifold $v=1 \Rightarrow \frac{dy}{dx}=1 \Rightarrow y=x+c$ is an exact solution.
The dynamics on the slow manifold corresponds to an outer solution.
To glue the two dynamics together we need a transition layer.




Outer Solution

$y=y_0+\varepsilon y_1+\varepsilon^2 y_2+\dots$

$O(1)$:

$y_0 y_0' - y_0 = 0$

$\frac{1}{2}\frac{d}{dx}(y_0'^2) = y_0$

$\Rightarrow y_0 = 0$ or $y_0 = x+a$

We get two outer solutions

$y_{left} = x+1$

$y_{right} = x-2$

Inner Solution

$$X = \frac{x - x_0}{\varepsilon^{\alpha}}$$

$$\Rightarrow \varepsilon^{1-2\alpha} \frac{d^2 y}{dX^2} = \varepsilon^{-\alpha} y \frac{dy}{dX} - y$$

Balancing we have

$$1 - 2\alpha = -\alpha$$

$$\Rightarrow \alpha = 1.$$

Expand

$$y = y_0 + \varepsilon y_1 + \dots$$

$O(1)$:

$$\frac{d^2 y_0}{dX^2} = y_0 \frac{dy_0}{dX}$$

$$\Rightarrow \frac{dy_0}{dX} = \frac{1}{2} y_0^2 + A$$

$$\Rightarrow \frac{1}{\frac{1}{2}y_0^2 + A} \frac{dy_0}{dX} = 1$$

$$\Rightarrow \int \frac{1}{\frac{1}{2}y_0^2 + A} dy_0 = X + C$$

$$\Rightarrow Y = A \tanh\left(\frac{AX}{2} + C\right)$$

$$= A \frac{\exp(AX/2 + C) - \exp(-AX/2 - C)}{\exp(AX/2 + C) + \exp(-AX/2 - C)}$$

$$= A\left(\frac{1 - Be^{-AX}}{1 + Be^{AX}}\right)$$

$$\lim_{X \to \infty} Y(X) = \lim_{x \to x_0} y_{right}(x) \text{ and } \lim_{X \to -\infty} Y(X) = \lim_{x \to x_0} y_{left}(x)$$

Therefore,

$$A = x_0 - 2$$
$$-A = x_0 + 1$$
$$\Rightarrow x_0 = 1/2, \; A = -3/2$$

Therefore,

$$Y_0 = -\frac{3}{2}\tanh\left(\frac{-3}{4}\left(\frac{x - 1/2}{\varepsilon}\right) + C\right)$$

From the phase portrait and symmetry in the problem we have that when $x = 1/2$, $Y_0 = 0$

$$\Rightarrow Y_0 = -\frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{x - 1/2}{\varepsilon}\right)\right)$$

Composite Expansion

We really need to do this in two parts

$$y \sim \begin{cases} x + 1 - \frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{x - 1/2}{\varepsilon}\right)\right) - \frac{3}{2}, & x < 1/2 \\ x - 2 - \frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{x - 1/2}{\varepsilon}\right)\right) + \frac{3}{2}, & x > 1/2 \end{cases}$$

$$\Rightarrow \boxed{y \sim x - \frac{1}{2} - \frac{3}{2}\tanh\left(\frac{3}{4}\left(\frac{x - 1/2}{\varepsilon}\right)\right)}$$