9

The Poincaré-Bendixson Theorem

The Poincaré-Bendixson theorem gives us a complete determination of the asymptotic behavior of a large class of flows on the plane, cylinder, and two-sphere. It is remarkable in that it assumes no detailed information about the vector field, only uniqueness of solutions, properties of ω limit sets, and some properties of the geometry of the underlying phase space. We begin by setting the framework and giving some preliminary definitions.

We will consider $\mathbf{C}^r$, $r \geq 1$, vector fields

$$\begin{aligned} \dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \qquad (x,y) \in \mathcal{P}, \end{aligned}$$

where $\mathcal{P}$ denotes the phase space, which may be the plane, cylinder, or two-sphere. We denote the flow generated by this vector field by

$$\phi_t(\cdot),$$

where the "·" in this notation denotes a point $(x,y) \in \mathcal{P}$.

The following definition will be useful.

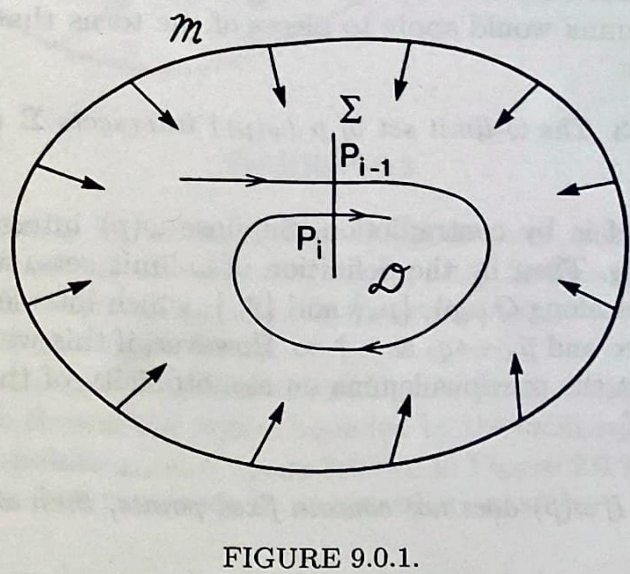

FIGURE 9.0.1.

Definition 9.0.1 *Let Σ be a continuous, connected arc in $\mathcal{P}$. Then Σ is said to be* transverse *to the vector field on $\mathcal{P}$ if the vector dot product of the unit normal at each point on Σ with the vector field at that point is not zero and does not change sign on Σ. Or equivalently, since the vector field is $\mathbf{C}^r$, $r \geq 1$, the vector field has no fixed points on Σ and is never tangent to Σ.*

Now we are in a position to actually prove the Poincaré-Bendixson theorem. We will first prove several lemmas from which the theorem will follow easily. Our presentation follows closely Palis and de Melo [1982]. In all that follows, $\mathcal{M}$ is understood to be a positively invariant compact set in $\mathcal{P}$. For any point $p \in \mathcal{P}$, we will denote the orbit of p under the flow $\phi_t(\cdot)$ *for positive times* $O_+(p)$ (also called the positive semiorbit of p).

Lemma 9.0.2 *Let $\Sigma \subset \mathcal{M}$ be an arc transverse to the vector field. The positive orbit through any point $p \in \mathcal{M}$, $O_+(p)$, intersects Σ in a monotone sequence; that is, if p_i is the i^{th} intersection of $O_+(p)$ with Σ, then $p_i \in [p_{i-1}, p_{i+1}]$.*

Proof: Consider the piece of the orbit $O_+(p)$ from p_{i-1} to p_i along with the segment $[p_{i-1}, p_i] \subset \Sigma$ (see Figure 9.0.1). (Note: of course, if $O_+(p)$ intersects Σ only once then we are done.)

This forms the boundary of a positively invariant region $\mathcal{D}$. Hence, $O_+(p_i) \subset \mathcal{D}$, and therefore we must have p_{i+1} (if it exists) contained in $\mathcal{D}$. Thus we have shown that $p_i \in [p_{i-1}, p_{i+1}]$. □

We remark that Lemma 9.0.2 does not apply immediately to toroidal phase spaces. This is because the piece of the orbit from p_{i-1} to p_i along with the segment $[p_{i-1}, p_i] \subset \Sigma$ needs to divide $\mathcal{M}$ into two "disjoint pieces." This would not be true for orbits completely encircling a torus. However, the lemma would apply to pieces of the torus that behave as $\mathcal{M}$ described above.

Corollary 9.0.3 *The ω-limit set of p ($\omega(p)$) intersects Σ in at most one point.*

Proof: The proof is by contradiction. Suppose $\omega(p)$ intersects Σ in two points, q_1 and q_2. Then by the definition of ω-limit sets, we can find sequences of points along $O_+(p)$, $\{p_n\}$ and $\{\bar{p}_n\}$, which intersect Σ such that $p_n \longrightarrow q_1$ as $n \uparrow \infty$ and $\bar{p}_n \longrightarrow q_2$ as $n \uparrow \infty$. However, if this were true, then it would contradict the previous lemma on monotonicity of the intersections of $O_+(p)$ with Σ. □

Lemma 9.0.4 *If $\omega(p)$ does not contain fixed points, then $\omega(p)$ is a closed orbit.*

Proof: The strategy is to choose a point $q \in \omega(p)$, show that the orbit of q is closed, and then show that $\omega(p)$ is the same as the orbit of q.

Choose $x \in \omega(q)$; then x is not a fixed point, since $\omega(p)$ closed and is a union of orbits containing no fixed points. Construct an arc transverse to the vector field at x (call it Σ). Now $O_+(q)$ intersects Σ in a monotone sequence, $\{q_n\}$, with $q_n \to x$ as $n \uparrow \infty$, but since $q_n \in \omega(p)$, by the previous corollary we must have $q_n = x$ for all n. Since $x \in \omega(q)$, the orbit of q must be a closed orbit.

It only remains to show that the orbit of q and $\omega(p)$ are the same thing. Taking a transverse arc, Σ, at q, we see by the previous corollary that $\omega(p)$ intersects Σ only at q. Since $\omega(p)$ is a union of orbits, contains no fixed points, and is connected, we know that $O(q) = \omega(p)$. □

Lemma 9.0.5 *Let p_1 and p_2 be distinct fixed points of the vector field contained in $\omega(p)$, $p \in \mathcal{M}$. Then there exists at most one orbit $\gamma \subset \omega(p)$ such that $\alpha(\gamma) = p_1$ and $\omega(\gamma) = p_2$. (Note: by $\alpha(\gamma)$ we mean the α limit set of every point on γ; similarly for $\omega(\gamma)$.)*

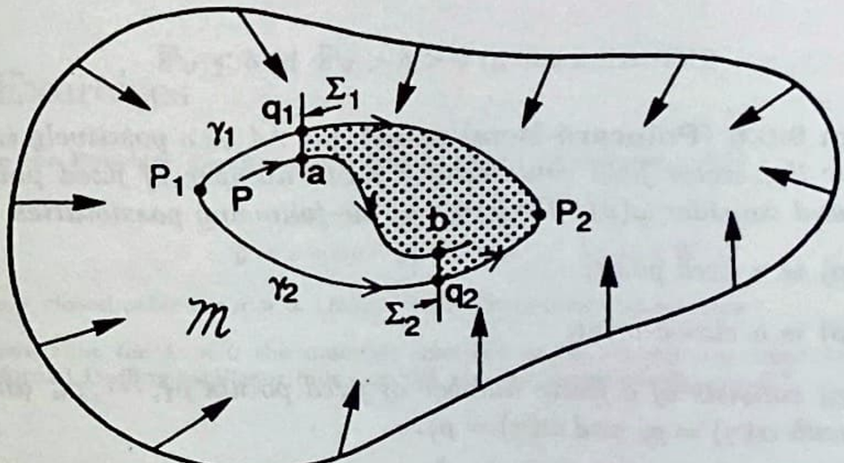

FIGURE 9.0.2.

Proof: The proof is by contradiction. Suppose there exist two orbits γ_1, $\gamma_2 \in \omega(p)$ such that $\alpha(\gamma_i) = p_1$, $\omega(\gamma_i) = p_2$, $i = 1, 2$. Choose points $q_1 \in \gamma_1$ and $q_2 \in \gamma_2$ and construct arcs Σ_1, Σ_2 transverse to the vector field at each of these points (see Figure 9.0.2).

Since $\gamma_1, \gamma_2 \subset \omega(p)$, $O_+(p)$ intersects Σ_1 in a point a and later intersects Σ_2 in a point b. Hence, the region bounded by the orbit segments and arcs connecting the points q_1, a, b, q_2, p_2 (shown in Figure 9.0.2) is a positively invariant region, but this leads to a contradiction, since $\gamma_1, \gamma_2 \subset \omega(p)$. □

Now we can finally prove the theorem.

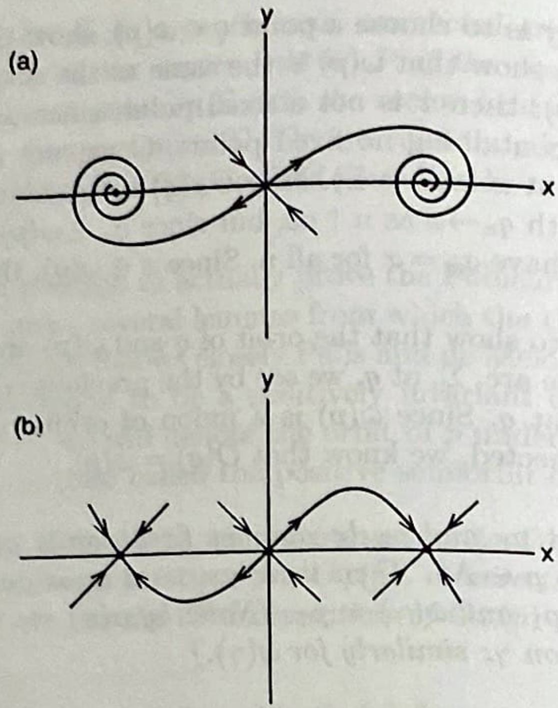

FIGURE 9.0.3. a) $0 < \delta < \sqrt{8}$; b) $\delta \geq \sqrt{8}$.

Theorem 9.0.6 (Poincaré-Bendixson) *Let $\mathcal{M}$ be a positively invariant region for the vector field containing a finite number of fixed points. Let $p \in \mathcal{M}$, and consider $\omega(p)$. Then one of the following possibilities holds.*

i) *$\omega(p)$ is a fixed point;*

ii) *$\omega(p)$ is a closed orbit;*

iii) *$\omega(p)$ consists of a finite number of fixed points $p_1, \cdots, p_n$ and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$.*

Proof: If $\omega(p)$ contains only fixed points, then it must consist of a unique fixed point, since the number of fixed points in $\mathcal{M}$ is finite and $\omega(p)$ is a connected set.

If $\omega(p)$ contains no fixed points, then, by Lemma 9.0.4, it must be a closed orbit. Suppose that $\omega(p)$ contains fixed points and nonfixed points (sometimes called regular points). Let γ be a trajectory in $\omega(p)$ consisting of regular points. Then $\omega(\gamma)$ and $\alpha(\gamma)$ must be fixed points since, if they were not, then, by Lemma 9.0.4, $\omega(\gamma)$ and $\alpha(\gamma)$ would be closed orbits, which is absurd, since $\omega(p)$ is connected and contains fixed points.

We have thus shown that every regular point in $\omega(p)$ has a fixed point for an α and ω limit set. This proves iii) and completes the proof of the Poincaré-Bendixson theorem. □

For an example illustrating the necessity of a finite number of fixed points in the hypotheses of Theorem 9.0.6 see Palis and de Melo [1982]. For generalizations of the Poincaré-Bendixson theorem to arbitrary closed two-manifolds see Schwartz [1963].

Example 9.0.1 (Application to the Unforced Duffing Oscillator). We now want to apply the Poincaré-Bendixson theorem to the unforced Duffing oscillator which, we recall, is given by

$$\dot{x} = y,$$
$$\dot{y} = x - x^3 - \delta y, \qquad \delta > 0.$$

Using the fact that the level sets of $V(x, y) = y^2/2 - x^2/2 + x^4/4$ bound positively invariant sets for $\delta > 0$, we see that the unstable manifold of the saddle must fall into the sinks as shown in Figure 9.0.3. The reader should convince him- or herself that Figure 9.0.3 is rigorously justified based on analytical techniques developed in this chapter. Note that we have not proved anything about the global behavior of the stable manifold of the saddle. Qualitatively, it behaves as in Figure 9.0.4, but, we stress, this has not been rigorously justified.

End of Example 9.0.1

9.1 Exercises

1. Use the Poincaré–Bendixson theorem to show that the vector field

$$\dot{x} = \mu x - y - x(x^2 + y^2),$$
$$\dot{y} = x + \mu y - y(x^2 + y^2), \qquad (x, y) \in \mathbb{R}^2,$$

has a closed orbit for $\mu > 0$. (*Hint:* transform to polar coordinates.)

2. Prove that for $\delta > 0$ the unstable manifold of the saddle-type fixed point of the unforced Duffing oscillator falls into the sinks as shown in Figure 9.0.3.

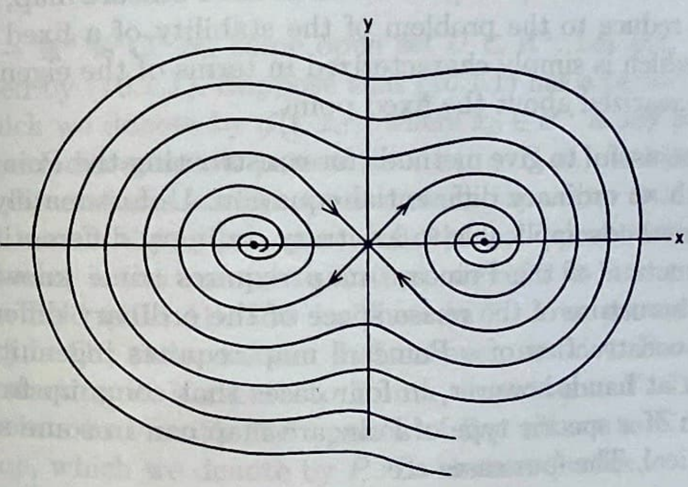

FIGURE 9.0.4.