Dual Immaculate Quasisymmetric Functions Expand Positively into Young Quasisymmetric Schur Functions

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FPSAC 2016

July 5, 2016
Some Schur-like Bases of $QSym$ and $NSym$
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Young Quasisymmetric Schur Functions

$QSym$ $\leftarrow$ $Sym$ $\rightarrow$ $NSym$

Schur Functions
Some Schur-like Bases of $QSym$ and $NSym$
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- Dual Immaculate Quasisymmetric Functions
- Young Quasisymmetric Schur Functions
- Immaculate Functions Young Noncommutative Schur Functions

$QSym \leftrightarrow Sym \leftrightarrow NSym$
Some Schur-like Bases of \( QSym \) and \( NSym \)

**\( QSym \):**

**Dual Immaculate Quasisymmetric functions:** \( \{ dI_\alpha \}_\alpha \)  
(Berg, Bergeron, Saliola, Serrano, Zabrocki, 2012)
- Generated by immaculate tableaux.
- Decompose positively into the fundamental basis.

**Young Quasisymmetric Schur functions:** \( \{ YQS_\alpha \}_\alpha \)  
(Luoto, Mykytiuk, and van Willigenburg, 2013)
- Generated by Young composition tableaux.
- Decompose positively into the fundamental basis.

**Question:** How are these bases related?
Some Schur-like Bases of $QSym$ and $NSym$

$NSym$:

Immaculate Functions $\{I_{\alpha}\}_{\alpha}$

- Can be defined using noncommutative Bernstein operators.
- Can be defined using a noncommutative analogue of the Jacobi-Trudi identity.

Young noncommutative Schur functions $\{\hat{s}_{\alpha}\}_{\alpha}$

- Littlewood-Richardson rule,

$$\hat{s}_{\alpha}\hat{s}_{\beta} = \sum C_{\alpha,\beta}^{\gamma} \hat{s}_{\gamma}$$

where $C_{\alpha,\beta}^{\gamma}$ counts certain tableaux.

Question: How are these bases related?
Main Theorem

Theorem
Let $dI_{\alpha}$ be the dual immaculate function indexed by $\alpha$ and let $YQS_{\beta}$ be the Young quasisymmetric Schur function indexed by $\beta$. Then

$$dI_{\alpha} = \sum_{\beta} c_{\alpha,\beta} YQS_{\beta}$$

where $c_{\alpha,\beta}$ is the number of DIRTs of shape $\beta$ and row strip shape $\alpha^{\text{rev}}$.

We use an insertion algorithm to prove this result.
Compositions

To each subset of \([n - 1]\), we associate a composition by sending the set \([a_1, a_2, \ldots, a_k]\) with \(a_1 < a_2 < \cdots < a_k\) to the composition \((a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k)\).

For example, if \(n = 4\),

\[
\text{comp}\left(\{1, 3\}\right) = (1, 2, 1).
\]

Let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n\). The Young composition diagram of shape \(\alpha\) is the collection of left-justified cells such that row \(i\) has \(\alpha_i\) cells. We will use French notation. The diagram below has shape \((2, 4, 3)\) and the cell with an X is in position \((3, 2)\).
Standard Immaculate Tableaux

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$. A filling of the Young composition diagram of shape $\alpha$ with the numbers $1, 2, \ldots, n$ is called a standard immaculate tableau if

1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right.
Standard Immaculate Tableaux

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n \). A filling of the Young composition diagram of shape \( \alpha \) with the numbers 1, 2, \ldots, \( n \) is called a standard immaculate tableau if

1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right.

Example

\[
\begin{array}{cccc}
3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{array}
\]

The five standard immaculate tableaux of shape (2, 4).
Dual Immaculate Quasisymmetric Functions

The descent set of a standard immaculate tableau $T$ is given by

$$\text{Des}_{dl}(T) = \{i \mid i + 1 \text{ is in a row strictly above } i \text{ in } T\}$$

\[
\begin{array}{ccc}
3 & 4 & 5 & 6 \\
1 & 2 & & \\
\end{array} & \begin{array}{ccc}
2 & 4 & 5 & 6 \\
1 & 3 & & \\
\end{array} & \begin{array}{ccc}
2 & 3 & 5 & 6 \\
1 & 4 & & \\
\end{array} \\
\begin{array}{ccc}
2 & 3 & 4 & 6 \\
1 & 5 & & \\
\end{array} & \begin{array}{ccc}
2 & 3 & 4 & 5 \\
1 & 6 & & \\
\end{array}
\]
Dual Immaculate Quasisymmetric Functions

The descent set of a standard immaculate tableau $T$ is given by

$$\text{Des}_{dl}(T) = \{ i \mid i + 1 \text{ is in a row strictly above } i \text{ in } T \}$$

For a composition $\alpha \models n$, the dual immaculate quasisymmetric function $dl_{\alpha}$ expands into the fundamental basis as

$$dl_{\alpha} = \sum_{T \text{ comp}} F_{\text{comp}(\text{Des}_{dl}(T))}$$

where the sum is over all standard immaculate tableaux of shape $\alpha$. So

$$dl_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)}.$$
Standard Young Composition Tableaux

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n \). A filling of the Young composition diagram of shape \( \alpha \) with the numbers 1, 2, \ldots, \( n \) is called a standard Young composition tableau if

1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right and
3. (YCT triple rule) for every subarray as below, if \( a > b \) then \( a > c \), where if \( c \) is empty, \( c = \infty \).

\[
\begin{array}{|c|c|}
\hline
b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|}
\hline
a \\
\hline
\end{array}
\]

Note that every standard Young composition tableau is a standard immaculate tableau.
Standard Young Composition Tableaux

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n \). A filling of the Young composition diagram of shape \( \alpha \) with the numbers 1, 2, \ldots, \( n \) is called a standard Young composition tableau if

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\[
\begin{array}{c|c}
  b & c \\
  \hline
  a \\
\end{array}
\]

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Standard Young Composition Tableaux

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$. A filling of the Young composition diagram of shape $\alpha$ with the numbers $1, 2, \ldots, n$ is called a standard Young composition tableau if

1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right and
3. (YCT triple rule) for every subarray as below, if $a > b$ then $a > c$, where if $c$ is empty, $c = \infty$.

$$
\begin{array}{c|c}
| b & c \\
| a & \\
\end{array}
$$

Note that every standard Young composition tableau is a standard immaculate tableau.

All except the second tableau are a standard Young composition tableaux.
Young Quasisymmetric Schur Function

The descent set of a standard Young Composition Tableau $T$ is given by

$$\text{Des}_{\mathcal{YQS}}(T) = \{i \mid i + 1 \text{ is weakly left of } i \text{ in } T\}.$$
Young Quasisymmetric Schur Function

The descent set of a standard Young Composition Tableau $T$ is given by

$$\text{Des}_{YQS}(T) = \{i \mid i + 1 \text{ is weakly left of } i \text{ in } T\}.$$ 

It is important to note that the two definitions of descent set for the same filling do not necessarily agree.
For a composition \( \alpha \vdash n \), the Young quasisymmetric Schur function \( YQS_\alpha \) expands into the fundamental basis as

\[
YQS_\alpha = \sum_T F_{\text{comp}}(\text{Des}_{YQS}(T))
\]

where the sum is over all standard Young composition tableaux of shape \( \alpha \).
Young Quasisymmetric Schur Function

For a composition $\alpha \models n$, the Young quasisymmetric Schur function $YQS_\alpha$ expands into the fundamental basis as

$$YQS_\alpha = \sum_T F_{\text{comp}(\text{Des}_{YQS}(T))}$$

where the sum is over all standard Young composition tableaux of shape $\alpha$.

So

$$YQS_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)}$$
Insertion Algorithm

Given a standard immaculate tableau, we obtain its reading word by reading left to right starting in the top row and working down.

We then insert this word to get a pair \((P, Q)\) where \(P\) is a standard Young composition tableau and \(Q\) is a recording tableau.
We insert the 2 into an empty filling.

\[ \text{Des}_I(T) = \{ i \mid i + 1 \text{ is in a row strictly above } i \} \]

\[ \text{Des}_YQS(T) = \{ i \mid i + 1 \text{ is weakly left of } i \} \]
Insertion Example

We insert the 2 into an empty filling.
Insertion Example

The 3 is placed next to the 2.
The 5 is placed next to the 3.
The 6 is placed next to the 5.
Insertion Example

\[
\begin{array}{cccc}
2 & 3 & 5 & 6 \\
1 & 4 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
2 & 3 & 5 & 6 \\
1 & 4 \\
\end{array}
\quad \rightarrow \quad
\begin{pmatrix}
\begin{array}{cccc}
2 & 3 & 5 & 6 \\
1 \\
\end{array}, &
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 \\
\end{array}
\end{pmatrix}
\]

Since 1 is smaller than all other elements, it starts a new row.
The 4 bumps the 5 and the insertion continues with the 5.
Insertion Example

The 4 bumps the 5 and the insertion continues with the 5.
The 5 is placed next to the 1.
## Insertion Example

\[
\begin{array}{cccc}
2 & 3 & 5 & 6 \\
1 & 4 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
2 & 3 & 5 & 6 \\
1 & 4 & & \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
2 & 3 & 4 & 6 \\
1 & 5 & & \\
\end{array},
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & & \\
\end{array}
\]

\[
\text{Des}_{dl}(T) = \{ i \mid i + 1 \text{ is in a row strictly above } i \text{ in } T \}
\]

\[
\text{Des}_{YQS}(T) = \{ i \mid i + 1 \text{ is weakly left of } i \text{ in } T \}
\]
## Insertion Example

<table>
<thead>
<tr>
<th>Immaculate Tableaux</th>
<th>Young Composition Tableaux</th>
<th>Recording Tableaux</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 4 5 6 1 2</td>
<td>3 4 5 6 1 2</td>
<td>1 2 3 4 5 6</td>
</tr>
<tr>
<td>2 4 5 6 1 3</td>
<td>2 3 5 6 1 4</td>
<td>1 2 3 4 5 6</td>
</tr>
<tr>
<td>2 3 5 6 1 4</td>
<td>2 3 4 6 1 5</td>
<td>1 2 3 4 5 6</td>
</tr>
<tr>
<td>2 3 4 6 1 5</td>
<td>2 3 4 5 1 6</td>
<td>1 2 3 4 5 6</td>
</tr>
<tr>
<td>2 3 4 5 1 6</td>
<td></td>
<td>1 2 3 4 5 6</td>
</tr>
</tbody>
</table>
### Insertion Example

<table>
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<th>Recording Tableaux</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Immaculate Tableaux" /></td>
<td><img src="image2" alt="Young Composition Tableaux" /></td>
<td><img src="image3" alt="Recording Tableaux" /></td>
</tr>
</tbody>
</table>

\[
d I(2,4) = \left( F(2,4) + F(1,2,3) + F(1,3,2) + F(1,4,1) \right) + F(1,5) \]

\[
= YQS(2,4) + YQS(1,5)
\]
Insertion Example

Immaculate Tableaux  Young Composition Tableaux  Recording Tableaux

\[
dl_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)}
\]
### Insertion Example

<table>
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<tr>
<td>\begin{tabular}{c</td>
<td>c</td>
<td>c</td>
</tr>
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\[ dl_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)} \]
\[ = YQS_{(2,4)} + \]
**Insertion Example**

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</tr>
<tr>
<td>1 2</td>
<td>1 2</td>
<td>5 6</td>
</tr>
<tr>
<td>2 4 5 6</td>
<td>2 3 5 6</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1 3</td>
<td>1 4</td>
<td>5 6</td>
</tr>
<tr>
<td>2 3 5 6</td>
<td>2 3 4 6</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1 4</td>
<td>1 5</td>
<td>5 6</td>
</tr>
<tr>
<td>2 3 4 6</td>
<td>2 3 4 5</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1 5</td>
<td>1 6</td>
<td>5 6</td>
</tr>
<tr>
<td>2 3 4 5</td>
<td>2 3 4 5 6</td>
<td>1 2 3 4 6</td>
</tr>
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<td>1</td>
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\[
d l_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)}
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</tr>
<tr>
<td>1 2</td>
<td>1 2</td>
<td>5 6</td>
</tr>
<tr>
<td>2 4 5 6</td>
<td>2 3 5 6</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1 3</td>
<td>1 4</td>
<td>5 6</td>
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<tr>
<td>2 3 5 6</td>
<td>2 3 4 6</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>1 4</td>
<td>1 5</td>
<td>5 6</td>
</tr>
<tr>
<td>2 3 4 6</td>
<td>2 3 4 5</td>
<td>1 2 3 4</td>
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<td>1 5</td>
<td>1 6</td>
<td>5 6</td>
</tr>
<tr>
<td>2 3 4 5</td>
<td>2 3 4 5 6</td>
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</tr>
<tr>
<td>1 6</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

\[
dl_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)} = YQS_{(2,4)} + YQS_{(1,5)}.
\]
Row Strips and Row Strip Shapes

The recording tableaux obtained by inserting the reading word of an immaculate tableau has a certain form. To describe the properties, we need two definitions.

Let $Q$ be a filling of a Young composition diagram with distinct entries $\{1, 2, \ldots, n\}$. A row strip of length $k$ is a maximal sequence of $k$ consecutive integers such that for all $1 \leq i < k$ in the row strip, $i + 1$ appears strictly right of $i$ in $Q$.

The row strip shape is the composition $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ where $\alpha_i$ is the length of the row strip which starts with the number $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + 1$.

Example
The filling below has row strip shape $(2, 4, 3)$

\[
\begin{array}{cccccc}
1 & 2 & 8 & 3 & 4 & 5 \\
7 & 6 & 9 & 3 & 4 & 5 \\
\end{array}
\]
Dual Immaculate Recording Tableau (DIRT)

If $Q$ is a recording tableau obtained from inserting the reading word of a standard immaculate tableau of shape $\alpha$ then the following hold for $Q$.

1. The rows of $Q$ increase from left to right.
2. The leftmost column of $Q$ increases from top to bottom.
3. The row strips start in the leftmost column of $Q$ and the row strip shape is $\alpha^{\text{rev}}$.
4. (Recording Triple Rule) Whenever the following subarray appears in $Q$ with $a > b$, we also have $a > c$ where if $c$ is empty, $c = \infty$.

$$\begin{array}{c}
a \\
| b | c \\
\end{array}$$

Such tableaux are called dual immaculate recording tableaux (DIRT).
Main Theorem

Insertion gives a descent preserving bijection between standard immaculate tableaux of shape $\alpha$ and pairs $(P, Q)$ where

- $P$ is standard Young composition tableau.
- $Q$ is a DIRT with row strip shape $\alpha^{rev}$.
- $P$ and $Q$ have the same shape.
Main Theorem

Insertion gives a descent preserving bijection between standard immaculate tableaux of shape $\alpha$ and pairs $(P, Q)$ where

- $P$ is standard Young composition tableau.
- $Q$ is a DIRT with row strip shape $\alpha^{rev}$.
- $P$ and $Q$ have the same shape.

Theorem

Let $dl_\alpha$ be the dual immaculate function indexed by $\alpha$ and let $YQS_\beta$ be the Young quasisymmetric Schur function indexed by $\beta$. Then

$$dl_\alpha = \sum_\beta c_{\alpha, \beta} YQS_\beta$$

where $c_{\alpha, \beta}$ is the number of DIRTs of shape $\beta$ and row strip shape $\alpha^{rev}$. 
An algorithm to find DIRTs with fixed row strip shape

We can build a rooted tree to find all the DIRTs of row strip shape \((\alpha_1, \alpha_2, \ldots, \alpha_k)\). For \(1 \leq m \leq k\), the nodes of the tree at level \(m\) are DIRTs of row strip shape \((\alpha_1, \alpha_2, \ldots, \alpha_m)\).
Finding DIRTs of row strip shape \((1, 2, 3)\)

DIRT\(_1\)s of row strip shape \((1, , )\)
Finding DIRTs of row strip shape \((1, 2, 3)\)
Finding DIRTs of row strip shape \((1, 2, 3)\)
Finding DIRTs of row strip shape \((1, 2, 3)\)

\[
dl_{(3,2,1)} = YQS_{(3,2,1)} + YQS_{(2,3,1)} + YQS_{(1,4,1)} + YQS_{(3,1,2)} + YQS_{(1,3,2)} + YQS_{(2,1,3)} + YQS_{(1,2,3)} + YQS_{(1,1,4)}.
\]
Decompositions in $NSym$

**Theorem**

Let $\hat{s}_\alpha$ be the Young noncommutative Schur function indexed by $\alpha$ and let $I_\beta$ be the immaculate function indexed by $\beta$. Then

$$\hat{s}_\alpha = \sum_\beta c_{\beta,\alpha} I_\beta$$

where $c_{\beta,\alpha}$ is the number of DIRTs of shape $\alpha$ and row strip shape $\beta^{\text{rev}}$. 

**Corollary**

We have the following.

1. $I_\alpha = \hat{s}_\alpha$ if and only if $\alpha$ is a partition.
2. For the hook shape $(1, n-k)$, we have

$$\hat{s}_{(1, n-k)} = \sum_{\beta \Vdash n} c_{\beta,\alpha} I_\beta$$
Decompositions in $NSym$

Theorem
Let $\hat{s}_\alpha$ be the Young noncommutative Schur function indexed by $\alpha$ and let $l_\beta$ be the immaculate function indexed by $\beta$. Then

$$\hat{s}_\alpha = \sum_{\beta} c_{\beta,\alpha} l_\beta$$

where $c_{\beta,\alpha}$ is the number of DIRTs of shape $\alpha$ and row strip shape $\beta^{\text{rev}}$.

Corollary
We have the following.

1. $l_\alpha = \hat{s}_\alpha$ if and only if $\alpha$ is a partition.
2. For the hook shape $(1^k, n-k)$, we have

$$\hat{s}_{(1^k, n-k)} = \sum_{\beta \models n \ell(\beta) = k+1} l_\beta.$$
Conjectures

Conjecture

Let $\alpha \vdash n$. If

$$YQS_\alpha = \sum b_{\alpha,\beta} dI_\beta$$

then $b_{\alpha,\beta} \in \{-1, 0, 1\}$. Moreover, for a fixed $\alpha$,

$$\sum_\beta b_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = (1^k, n-k), \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture

Let $\lambda$ be a partition of $n$ with $k$ parts all of which are distinct. Then

$$YQS_\lambda = \sum_{\pi \in S_k} (-1)^{\ell(\pi)} dI_{\pi(\lambda)}$$

where $\ell(\pi)$ is the length of the permutation $\pi$ and

$\pi(\lambda) = (\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(k)})$. 
THANK YOU!