Strongly n-trivial Knots

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Abstract

We prove that given any knot $k$ of genus $g$, $k$ fails to be strongly $n$-trivial for all $n$, $n \geq 3g - 1$.

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1 Introduction

We start with a little background.

Definition 1.1. A knot $k$ is called “$(n-1)$-trivial” if there exists a projection $k$, such that one can choose $n$ disjoint sets of crossings of the projection with the property that making the crossing changes corresponding to any of the $2^n - 1$ nontrivial combination of the sets of crossings turns the original knot into the unknot. The collection of sets of crossing changes is said to be an “$(n-1)$-trivializer for $k$”.

Conjecture 1.2. The unknot is the only knot that is $n$-trivial for all $n$.

Note: A knot that is $n$-trivial is automatically $m$-trivial for all $m \leq n$.

Work of Gusarov [Gu] and Ng and Stanford [NS] shows that this question equates to showing that the only knot with vanishing Vassiliev invariants for all $n$ is the unknot. Thus, Conjecture 1.2 is at the heart of the study of Vassiliev invariants.

One reason why this question is interesting is that it takes a geometric approach to Vassiliev invariants, instead of the traditional algebraic approach and therefore is relatively unexplored. Vassiliev invariants measure geometric
properties of knots, which in turn are geometric objects, so it is reasonable to hope that the geometry might play an integral role in their study.

The following definition derives its motivation from $n$-triviality.

**Definition 1.3.** A knot $k$ is called “strongly $(n-1)$-trivial” if there exists a projection of $k$, such that one can choose $n$ crossings of the projection with the property that making the crossing changes corresponding to any of the $2^n - 1$ nontrivial combinations of the selected crossings turns the original knot into the unknot. The collection of crossing changes is said to be a “strong $(n-1)$-trivializer for $k$”.

**Note:** The expression “$n$ adjacent to the unknot” is used interchangeably with “strongly $(n-1)$-trivial.” We will stick with the latter throughout this paper.

In Section 6 we show that for any $n$ there is a non-trivial knot that is strongly $n$-trivial. On the other hand in Section 5 we prove the main result of this paper:

**Theorem 1.4.** Any non-trivial knot $k$ of genus $g$ fails to be strongly $n$-trivial for all $n$, $n \geq 3g - 1$.

**Note:** A knot that is strongly $n$-trivial is automatically strongly $m$-trivial for all $m \leq n$. Also any knot that is strongly $n$-trivial is clearly $n$-trivial, too.

In analogy with Conjecture 1.2 we have

**Corollary 1.5.** The unknot is the only knot that is strongly $n$-trivial for all $n$.

Theorem 1.4 is proven by repeated use of the following theorem of Gabai

**Theorem 1.6.** (Corollary 2.4 [G]) Let $M$ be a Haken manifold whose boundary is a nonempty union of tori. Let $F$ be a Thurston norm minimizing surface representing an element of $H_2(M, \partial M)$ and let $P$ be a component of $\partial M$ such that $P \cap F = \emptyset$. Then with at most one exception (up to isotopy) $F$ remains norm minimizing in each manifold $M(\alpha)$ obtained by filling $M$ along an essential simple closed curve $\alpha$ in $P$. In particular $F$ remains incompressible in all but at most one manifold obtained by filling $P$. 

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2 Notation

Let $k$ be a knot that is strongly $(n-1)$-trivial. Let $p : k \to R^2$ be a projection with crossings $\{a_1, \ldots, a_n\}$ demonstrating the strong $(n-1)$-triviality. For each $a_i$ let $c_i$ be the small vertical circle that bounds a disk $D_i$ that intersects $k$ geometrically twice, but algebraically 0 times. We call the $c_i$ linking circles of $k$ and call $D_i$ a crossing disk after [ST]. Let $M$ be the link exterior of $k \cup c_1 \cup \ldots \cup c_n$ and $P_i$ be the torus boundary component in $M$ corresponding to $c_i$. Either +1 or −1 filling of $P_i$ will result in the desired crossing change depending on orientation. We adopt the convention that each $P_i$ will be oriented so that +1 filling of $P_i$ corresponds to the appropriate crossing change dictated by $a_i$.

3 Irreducibility

Lemma 3.1. Let $k$ be a nontrivial knot. Let $\{c_1, \ldots, c_n\}$ be linking circles for $k$ that show $k$ is strongly $(n-1)$-trivial, then $M$, the exterior of $k \cup c_1 \cdots \cup c_n$, is irreducible and $M$ is therefore Haken.

Proof. Assume $M$ is reducible. Let $S$ be a sphere that does not bound a ball on either side. $S$ cannot be disjoint from $D_1 \cup \ldots \cup D_n$ or else it would bound a ball on the side that does not contain $k$. Assume $S$ intersects $D_1 \cup \cdots \cup D_n$ minimally and transversally. The intersection will consist of a union of circles. Let $r$ be one of these circles that is innermost on $S$ (any circle that bounds a disk on $S$ disjoint from all the other circles of intersection). Without loss of generality assume $r \subset D_1$. $r$ cannot be trivial on $D_1 - (D_1 \cap k)$ since $S \cap D_1$ is minimal. $r$, however must be trivial in $M$ so must divide $D_1$ into two pieces, one containing both points of $D_1 \cap k$ and the other consisting of an annulus running from $r$ to $c_1$. This disk on $S$ bounded by $r$ shows that $c_1$ bounds a disk in the exterior of $k$. This, however, means that +1 surgery on $c_1$ leaves $k$ unchanged instead of turning it into an unknot, yielding the desired contradiction. \hfill \Box

4 Minimal genus Seifert surfaces

This section is dedicated to proving the following theorem.
Theorem 4.1. If $k$ has a strong $(n-1)$-trivializer $\{c_1, \ldots, c_n\}$ and $F$ is a Seifert surface for $k$ disjoint from $\{c_1, \ldots, c_n\}$ which is minimal genus among all such surfaces, then $F$ is a minimal genus Seifert surface for $k$.

Proof. Because the $c_i$ have linking number 0 with $k$ we can find a Seifert surface for $k$ disjoint from the $c_i$. Let $F$ be a minimal genus Seifert surface for $k$ in the link complement.

We supplement the notation introduced in Section 2. Recall $M$ is the link exterior of $k \cup c_1 \cup \cdots \cup c_n$. Let $L$ be the corresponding link of $n + 1$ components in $S^3$. $P_1$ is the torus boundary component in $M$ corresponding to $c_1$. Let $M(\alpha)$ refer to the manifold obtained by filling $M$ along an essential simple closed curve of slope $\alpha$ in $P_1$. When $\alpha = 1/m, m \in Z$, $M(\alpha)$ is a link exterior. Let $L_\alpha$ be the corresponding link in $S^3$. Let $k_\alpha$ be the image of $k$ in $L_\alpha$ and $F_\alpha$ be the image of $F$ in $L_\alpha$.

We now prove Theorem 4.1 by induction on $n$. If $F$ is ever a disk then Theorem 4.1 is clearly true, so we will assume that $F$ is not a disk throughout the proof.

The base case: Let $k$ be a strongly 0-trivial knot. This means that $k$ is unknotting number 1 and there is one linking circle $c_1$ that dictates a crossing change that unknots $k$.

By Lemma 3.1 if $M$ is reducible, then $k$ is the unknot. As in the proof of Lemma 3.1 $c_1$ bounds a disk in the complement of $k$, so $k \cup c_1$ is the unlink on two components. Therefore, $F$ being least genus must be a disk, which is a contradiction, verifying the claim for $M$ reducible and $n = 1$. We may assume $M$ is irreducible to complete the base case. $k_1$ is the unknot. Since $F_1$ is not a disk, it is no longer norm minimizing after the filling. Thus by Theorem 1.6 $F$ is norm minimizing under any other filling of $P_1$. In particular $F_\infty$ is Thurston norm minimizing for $L_\infty$, which is just $k$. Thus, $F$ is a least genus Seifert surface for $k$.

The inductive step: Now we assume that if $k$ has a strong $(n-2)$-trivializer $\{c_1, \ldots, c_{n-1}\}$ and $F$ is a Seifert surface for $k$ disjoint from $\{c_1, \ldots, c_{n-1}\}$, which is minimal genus among all such surfaces, then $F$ is also a minimal genus Seifert surface for $k$ and show that the same must be true for any strong $(n-1)$-trivializer for $k$.

Again by Lemma 3.1 if $M$ is reducible, $k$ must be the unknot. As in previous arguments, the separating sphere $S$ must intersect at least one $D_i$ in a curve that is essential on $D_i - k$. Without loss of generality, we may
assume that $D_n$ is such a disk. Then $c_n$ bounds a disk in the complement of $k \cup \{c_1, \ldots, c_{n-2}\}$. Since $\{c_1, \ldots, c_{n-1}\}$ forms a strong $(n-2)$-trivializer for $k$, the induction assumption implies $k$ bounds a disk $\Delta$ disjoint from $c_1 \cup \ldots \cup c_{n-1}$. Since $c_n$ bounds a disk disjoint from $k \cup c_1 \cup \ldots \cup c_{n-1}$, $\Delta$ can clearly be chosen to be disjoint from $c_n$, too, but this contradicts the assumption that $F$ was minimal genus, but not a disk.

We now may finish the proof of Theorem 4.1 knowing that $M$ is irreducible. $k_1$ is an unknot in the link $L_1$. $\{c_1, \ldots, c_{n-1}\}$ is a strong $(n-2)$-trivializer for $k$ in $L_1$. The inductive assumption means that $k_1$ bounds a disk in the exterior of $L_1$. This disk is in the same class as $F_1$ in $H_2(M(1), \partial M(1))$, thereby showing that $F_1$ is not Thurston norm minimizing. Thus, by Theorem 1.6 $F$ remains norm minimizing under any other filling of $P_n$. In particular $F_\infty$ is Thurston norm minimizing in $L_\infty$. Thus, $F$ is a least genus Seifert surface for $k$ in the complement of $\{c_1, \ldots, c_{n-1}\}$. $\{c_1, \ldots, c_{n-1}\}$, however, forms a strong $(n-2)$-trivializer for $k$ in $L_\infty$. By the inductive assumption, $F$ must be Thurston norm minimizing for $k$ in the knot complement as well as the link complement.

\[ \square \]

5 Arcs on a Seifert surface

We now prove Theorem 1.4: Any non-trivial knot $k$ of genus $g$ fails to be strongly $n$-trivial for all $n, n \geq 3g - 1$.

\textit{Proof.} Let $k$ be strongly $n$-trivial with $n$-trivializers $\{c_1, \ldots, c_{n+1}\}$. Let $F$ be a minimal genus Seifert surface for $k$ disjoint from $\{c_1, \ldots, c_{n+1}\}$ as in Theorem 4.1. $F$ has genus $g$.

Each linking circle $c_i$ bounds a disk $D_i$ that intersects $F$ in an arc running between the two points of $k \cap D_i$ and perhaps some simple closed curves. Simple closed curves inessential in $D_i - k$ can be eliminated since $F$ is incompressible. Any essential curves $s_j$ must be parallel to $c_i$ in $D_i - k$. These curves can be removed one at a time using the annulus running from $c_i$ to the outermost $s_j$ to reroute $F$, decreasing the number of intersections. Thus, if $F$ is assumed to have minimal intersection with each of the $D_i$ then it intersects each one in an arc which we shall call $a_i$ as in Figure 1. Each $a_i$ is essential in $F$. Otherwise $c_i$ would bound a disk disjoint from $F$ and the crossing change along $c_i$ would fail to unknot $k$.

\textbf{Lemma 5.1.} $a_i$ is never parallel on $F$ to $a_j$ for $i \neq j$.
Figure 1: A Seifert surface passing disjointly through a linking circle

Proof. If \( a_i \) is parallel on \( F \) to \( a_j \) there must be an annulus running from \( P_i \) to \( P_j \) in the link exterior. Recall that we adopted the convention that \( P_i \) and \( P_j \) are each oriented so that +1 surgery results in the appropriate crossing changes. The two tori cannot have opposite orientations or else +1 fillings on both \( P_i \) and \( P_j \) is the same as \( \infty \) fillings on both, thus, instead of unknotting \( k \) changing both crossings leaves \( k \) knotted. If the two tori have the same orientation we could replace \( P_i \cup P_j \) with a single torus \( T \) obtained by cutting and pasting of the two tori along the annulus. Now +1 filling for \( P_i \) and \( \infty \) filling for \( P_j \) is equivalent to +1 filling on \( T \), while +1 filling on both \( P_i \) and \( P_j \) is equivalent to \( \frac{1}{2} \) filling on \( T \). This implies that \( F \) fails to be norm-minimizing after both +1 and \( \frac{1}{2} \) filling of \( T \). This contradicts Theorem 1.6 completing the proof of the Lemma.

Then \( \{a_1, \ldots, a_n\} \) is a collection of embedded arcs on \( F \), no two of which are parallel. An Euler characteristic argument shows that \( m \leq 3g - 1 \). Since the arcs are in one to one correspondence with the linking circles, a strong \( n \)-trivializer can produce at most \( 3g - 1 \) linking circles for \( k \) finally proving Theorem 1.4.

We note that Theorem 1.4 predicts that a genus one knot can be at most strongly 1-trivial. Given standard projections of the trefoil and the figure eight knot it is easy to find a pair of crossing changes that show the knots are strongly 1-trivial. The theorem is therefore sharp at least for genus one

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knots. It is possible, but unlikely, that the theorem remains as precise for higher genus knots.

Finally as noted in the introduction, Theorem 1.4 implies Corollary 1.5: The unknot is the only knot that is strongly \( n \)-trivial for all \( n \).

6 Constructing strongly \( n \)-trivial knots

One might think that there exists a bound \( n \) such that no nontrivial knot is strongly \( n \)-trivial. Given any \( n \), this section gives one way to construct strongly \( n \)-trivial knots.

In Figure 3 we will give projections of graphs that show how to turn an unknot into a strongly \( n \)-trivial knot. The circle running around the outside of the graph should be viewed as an unknot \( k' \). Each arc \( a_i \) suggests a linking circle \( c_i \) and a crossing disk \( D_i \). If we alter the link \( k \cup c_1 \cup \cdots \cup c_n \) in \( S^3 \) by twisting \(-1\) times along each of the disks \( D_1, D_2, \ldots, D_n \), \( k' \) becomes a new knot \( k \) see Figure 2. The linking circles remain fixed, so we get a new link in \( S^3, k \cup c_1 \cdots \cup c_m \).

![Diagram](attachment:diagram.png)

Figure 2: A graph contains instructions for turning the unknot into a knot (or perhaps another unknot).

Figure 3 gives graphs that generate examples of strongly 1-trivial and strongly 2-trivial knots. Note that the figure on the right is obtained from the figure on the left by replacing one arc by two new arcs that follow along the original arc, clasp, return along the original arc, and then, close to the
boundary, clasp once again. This process could be iterated indefinitely by choosing an arc of the new graph and repeating the construction. It is modeled on doubling one component of a link. Given a Brunnian link of \( n \) components (a nontrivial link for which any \( n - 1 \) components is the unlink), doubling one of the components yields a Brunnian link of \( n + 1 \) components. The graph on the left in Figure 3 has a Brunnian link of 2 components as a subgraph and the one on the right has the double of that link as a subgraph. Let \( \Gamma_n \) be the graph after \( n - 2 \) iterations (\( n \geq 2 \)).

![Figure 3: Examples of crossing changes for the unknot that create nontrivial knots that are strongly 1-trivial (left) and strongly 2-trivial (right). Note that each contains a subgraph that is a Brunnian link of \( n + 1 \) components.](image)

**Theorem 6.1.** \( \Gamma_n \) contains a Brunnian link, \( l_{n+2} \), of \( n + 2 \) components and yields \( k \) a non-trivial, strongly \( (n+1) \)-trivial knot.

**Proof.** The link consists of the arcs \( \{a_1, \ldots, a_{n+2}\} \), together with short segments from \( k' \) connecting the end points of the segments (and disjoint from the end points of the other segments). The base case is trivial because, \( \Gamma_0 \) contains a Brunnian link of 2 components: the Hopf link. \( \Gamma_n \) is obtained from \( \Gamma_{n-1} \) by doubling one of the components of a Brunnian link of \( n + 1 \) components. This yields a Brunnian link of \( n + 2 \) components.

As a result of the Brunnian structure in \( \Gamma_n \) any \( n+1 \) edges from \( \{a_1, \ldots, a_{n+2}\} \) can be disjointly embedded on a disk bounded by \( k' \). So \( k' \) forms an unlink with any proper subset of \( \{c_1, \ldots, c_{n+2}\} \).

We can use this fact to show that \( \{c_1, \ldots, c_{n+2}\} \) are an \( n \)-trivializer for \( k \). Let \( S \) be any nontrivial subset of \( \{c_1, \ldots, c_{n+2}\} \). Let \( S^c \) be the complement of \( S \). If we take \( k \) together with \( S \), and do \( +1 \) surgery on each component of \( S \)

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the resulting knot is an unknot. This is because it is exactly the same as if we took \( k' \) and did \(-1\) surgery on each of the components of \( S^c \). Since \( S \) is a nontrivial subset, \( S^c \) is a proper subset. \( k' \) together with a the linking circles in \( S^c \), therefore form an unlink, so each of the components of \( S^c \) bounds a disk disjoint from \( k' \) and doing \(-1\) surgery on these linking circles leave \( k' \) unchanged.

Now that we know that \( k \) is strongly \((n + 1)\)-trivial, we need only show \( k \) is a non-trivial knot. Assume \( k \) is trivial. By Theorem 4.1, \( k \) bounds a disk \( \Delta \) in the complement of \( c_1 \cup \cdots \cup c_{n+2} \). Since \( k \cup c_1 \cup \cdots \cup c_{n+2} \) was obtained from \( k' \cup c_1 \cup \cdots \cup c_{n+2} \) by spinning along the \( D_i \)'s, the exteriors of the two links are homeomorphic, and therefore \( k' \) must bound a disk \( \Delta' \) also disjoint from \( c_1 \cup \cdots \cup c_{n+2} \) (note that one could even prove that both \( k \cup c_1 \cup \cdots \cup c_{n+2} \) and \( k' \cup c_1 \cup \cdots \cup c_{n+2} \) are unlinks). \( k' \) intersects each \( D_i \) in 2 points, so as before we may assume \( \Delta' \cap D_i \) is an arc for each \( i \), but these arcs must, of course, be isotopic to the \( a_i \)'s which in turn shows that the \( a_i \) can be disjointly embedded on \( \Delta' \), proving that \( l_{n+2} \) is planar and not Brunnian, the desired contradiction. Thus, \( k \) is a strongly \((n + 1)\)-trivial knot, but not the unknot.

\[ \square \]

7 References


[NS] Ng, Ka Yi; Stanford, Ted, *On Gusarov’s groups of knots*, to appear in Math Proc Camb Phil.