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CONVEX BRUNNIAN LINKS

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ABSTRACT

We prove that the Borromean Rings are the only Brunnian link of 3 or 4 components that can be built out of convex curves.

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Mathematics Subject Classification 2000: 57M02

1. Introduction

Definition 1.1. A Brunnian link is a link $L = k_1 \cup k_2 \cup \cdots \cup k_n$ of $n \geq 3$ components such that L is not an unlink, but every proper sublink of L is an unlink.

The most famous example of a Brunnian link is the Borromean rings, but there are an infinite number of different Brunnian links for any given n.

In this paper we bring to light a theorem of Freedman and Skora [2] about Brunnian Links and then generalize several cases of the theorem. In 1987 Freedman and Skora proved the following theorem in their paper [2].

Theorem 1.2 [2]. *No Brunnian Link can be built out of round circles.*

This is in some sense a surprising result since Fig. 1 appears to consist of three circles! The seeming contradiction is resolved by the realization that although the projection is made up of three circles, the link itself is not made up of circles. The projection yields an optical illusion. Viewing it from a different perspective would reveal that the "circles" must be bent. (This is a similar illusion to the one that allows the knot 5_1 to appear to be built out of five sticks even though, only the unknot can be built out of fewer than six sticks.)

The special case of the Borromean rings was reproven in the early 90's by Lindstrom and Zetterstrom in [3], and independently (but unpublished) in the

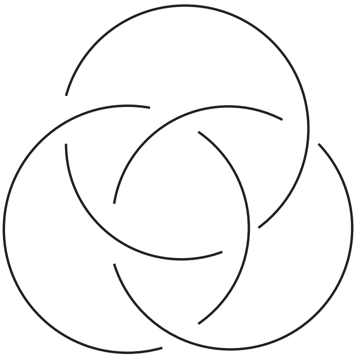

Fig. 1. The Borromean rings pictured here appear to be made out of circles.

simplest proof yet by Agol in 1993. Lindstrom and Zetterstrom apparently did not know about Freedman and Skora's proof when they published their more complicated and less general proof four years later.

Note that we can certainly form the Borromean rings out of two circles and an ellipse. For example in the xy plane take $L_1 = k_1 \cup k_2 \cup k_3$, consisting of the ellipse, k_1, $\frac{x^2}{4} + \frac{y^2}{25} = 1$ In the xz plane take the circle k_2, $x^2 + z^2 = 9$, and in the yz plane take the circle k_3, $y^2 + z^2 = 16$.

This paper generalizes some cases of Theorem 1.2 in the following way.

We ask, what happens if instead of requiring the components of the link to be round circles, we only require them to bound convex planar regions (we will call such a curve convex), and answer:

Theorem 1.3. *The Borromean rings are the unique Brunnian links of three or four components that can be formed out of convex curves.*

We now make a few observations about Brunnian links. Note that for each $n, n \geq 3$ there are an infinite number of Brunnian links with n components. One method of construction is to take the infinite family of non-trivial links made up of two unknotted components and take a Bing double of one of the components (see Fig. 2). This yields an infinite family of three component Brunnian links. If we double one of the components of the new links, we get an infinite family of four component Brunnian links. We can, of course, continue this process as long as we like.

It is obviously also true that any n component Brunnian link can be made out of $(n-1)$ circles and one more component, but by Theorem 1.2 this component will never be a circle. Also we observe the following lemma.

Lemma 1.4. *Although there are no convex planar Brunnian links of four components, for any $n \geq 3$ we can construct a Brunnian link of n components in which all the components are planar and all but one of the components are circles.*

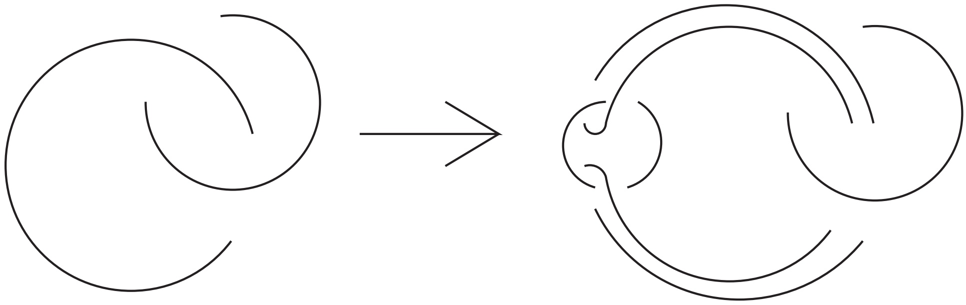

Fig. 2. The Bing double of the Hopf link yields a Brunnian link with planar components.

Proof. Build the Hopf link out of two circles and take the Bing double of one of the components, replacing it with one small circle and one planar curve as in Fig. 2. To get an n component link iterate this process $n-3$ more times doubling the single component that is not a circle each time. □

2. Convex Brunnian Links

We now turn to the proof of Theorem 1.3.

Theorem 1.3. *The Borromean rings are the unique Brunnian links of three or four components that can be formed out of convex curves.*

Proof. Note that a planar closed curve in R^3 bounds a uniquely determined flat disk. In general these disks can be disjoint, can intersect in a single ribbon singularity, or a single clasp singularity. Figure 3 shows each of the latter type of singularities. Figure 4 shows a picture of each of the disks drawn abstractly with the intersection with the other disk recorded on it.

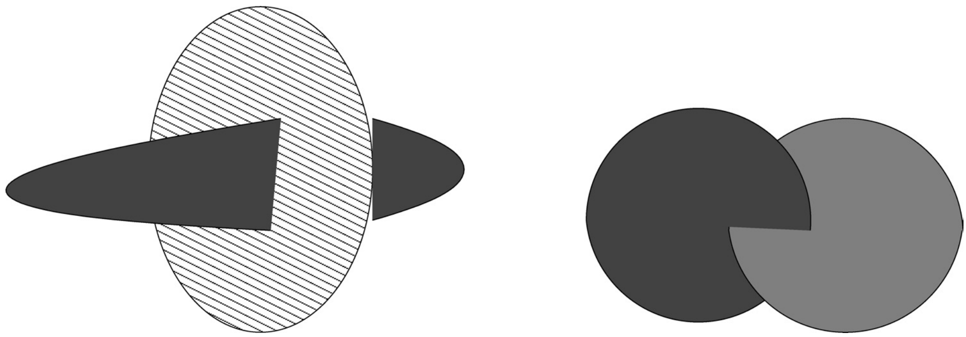

Fig. 3. A ribbon singularity and a clasp singularity.

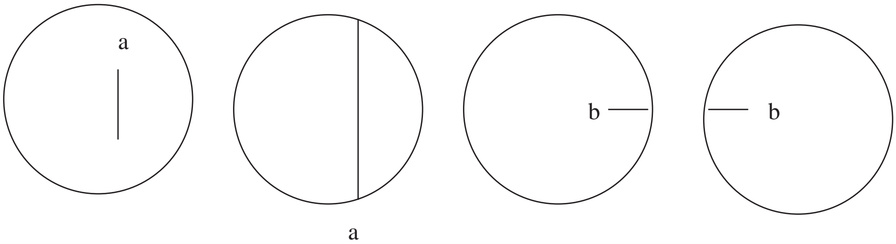

Fig. 4. An abstract picture of a ribbon and a clasp singularity. The first disk corresponds to the interior arc from the ribbon singularity, the second, the exterior arc. The third and fourth disks have the clasp singularities drawn.

In our context, because the disks are planar and convex, if we see a clasp singularity, the two knots must form a Hopf link and have non-trivial linking number and thus are not part of a Brunnian link or any link with pairwise linking numbers 0. Thus we may assume that we only see ribbon singularities. Note that every time we have a ribbon singularity on one disk we see an *exterior arc* that has both end points on the knot and is properly embedded on the disk, and on the other we see an *interior arc*, that is totally contained on the interior of the disk, including the end points, just as in Fig. 4.

Recall L_1 and L_2, the two links from the start of this section. Note that in L_1 each disk contains one arc of each type, and in L_2, the first disk contains two exterior arcs, the second has one exterior arc and one interior arc, and the final disk contains two interior arcs.

In both cases we say we have a triple point singularity since all three disks intersect in a single point. We say the diagram of intersections on one of the disks has a "t" in it if it has two edges intersecting in a triple point.

2.1. *The case of 3 convex components*

Let $L = s_1 \cup s_2 \cup s_3$ be a link consisting of three convex, planar curves, bounding planar disks Δ_1, Δ_2, and Δ_3 respectively. Assume we do not have the Borromean rings. We would like to prove that L is the unlink. If there are no triple points in the intersection, then each disk has one or two ribbon singularities on it. There are three ways to pair up the disks (and thus at most three intersections) so there are at most three interior arcs and at most three exterior arcs total on the three disks.

Lemma 2.1. *No component of a Brunnian link can bound a disk disjoint from the other components of the link, and thus no disk can have only exterior arcs for its intersection pattern.*

Proof. If one disk has only exterior arcs on it, then its boundary is a knot that bounds a disk that is disjoint from the other knots. The boundary of a small regular neighborhood of the disk is a sphere that separates one component of the link from the rest and thus we have a split link and not a Brunnian link. □

Possible intersection patterns for the disks in the four component case are depicted in Figs. 5–8. In each figure we draw Δ_1, $\Delta_2, \ldots, \Delta_n$, with Δ_1 in the top left corner, Δ_2 in the top row just to the right of Δ_1 on the left and $\Delta_{(i+1)}$ always directly to the right of Δ_i. Then the intersections are recorded by arcs and labeled with variables. In Fig. 8, for example, the intersection of Δ_1 with Δ_2 is the arc labeled a. Two disks intersect in the picture if and only if there is an arc on each with the same label. To distinguish between the image of a on Δ_1 and on Δ_2 we will call the former a_1 and the later a_2. The same convention will be used for all intersections — the subscript of the arc on D_i will be i, and a variable such as a will appear on exactly two disks — the two that intersect in the corresponding arc. Note also that a triple point can be identified by giving the three arcs that meet at it, such as a, b, d which is the triple point from Δ_1, Δ_2, and Δ_3 in Fig. 8. On Δ_2 that triple point is represented by the intersection of a_2 and d_2, on Δ_1 it is represented by the intersection of a_1 and b_1, and then on Δ_3 by the remaining possible pairing of a, b, and d (b_3 and d_3).

We say an exterior arc a_i for a disk Δ_i is "outermost" on Δ_i if one of the components of $D - a_i$ contains no exterior arcs or interior arcs. For example, in

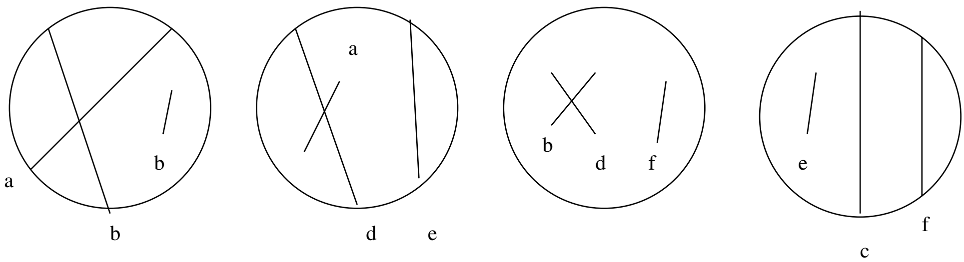

Fig. 5. Diagrams for one triple point. Δ_1 is on the left Δ_2 to its right, Δ_3 the third disk, and Δ_4 is on the right.

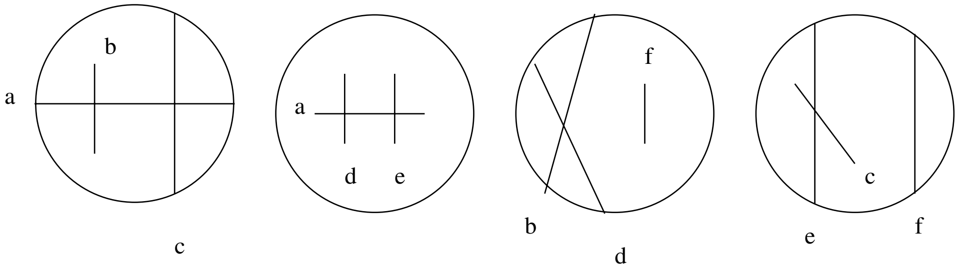

Fig. 6. Diagrams for two triple points.

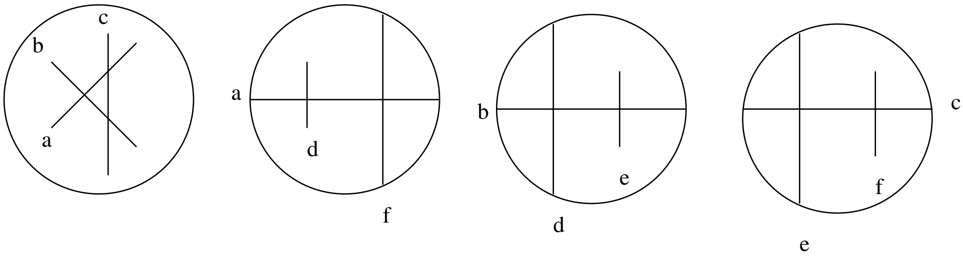

Fig. 7. Diagrams for three triple points.

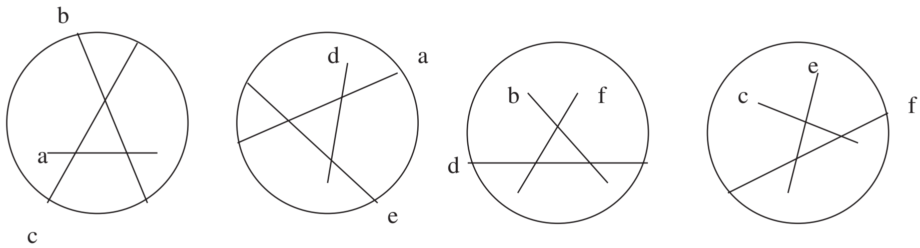

Fig. 8. Diagrams for four triple points.

Fig. 5 the arc e_2 on Δ_2 is outermost as is the arc f_4 on Δ_4. None of the other arcs in that picture are outermost (an arc containing a triple point is never considered to be outermost).

For the three component case, let us first examine the case where there are no triple points of intersection. If not all of the disks intersect each other then one disk must contain only exterior arcs, contradicting Lemma 2.1. Thus all disks must intersect each other and since by Lemma 2.1 no disk has only exterior arcs on it, each disk has one interior arc on it, and one exterior arc on it, and thus all exterior arcs are outermost arcs. Without loss of generality, assume that Δ_1 contains an outermost exterior arc corresponding to its intersection with Δ_2. Let S be a sphere that is the boundary of a small regular neighborhood of Δ_1. Since $\Delta_1 \cap \Delta_2$ was an outermost arc on Δ_1, $S - (S \cap \Delta_2)$ has a component T that is a disk disjoint from all of the Δ_i. We may alter Δ_2 by deleting the sub-disk cut off by $S \cap \Delta_2$ and replacing it with T to get a new disk Δ_2' that is now disjoint from Δ_1, decreasing the total number of interior arcs by one and the total number of exterior arcs by one, too. We will call this process, "pushing" Δ_2 along Δ_1 to decrease the number of intersections. Note that this means s_2 bounds a disk Δ_2' with no interior arcs on it, and by Lemma 2.1, we have a split link and thus cannot have a Brunnian link!

Thus to have a Brunnian link of three convex planar components we must see a triple point. Since there are three exterior arcs and three interior arcs, there are two options now. The first case is that on one disk there are two exterior arcs (and no interior arcs), on one disk there is one of each, and on the final disk there are two interior arcs (and no exterior arcs). In this case, the first disk shows that the link is not Brunnian by Lemma 2.1. The second case is that on each disk the triple point is part of one exterior arc and one interior arc. In this case, the disks lie in three planes intersecting in a point (since there is a triple point and the three disks are planar). Without loss of generality, we can assume that Δ_1 lies in the xy plane, Δ_2 lies in the xz plane, and Δ_3 lies in the yz plane (in reality, the planes may not be perpendicular, but that will have no effect on the crucial topological information in this argument). We can assume that $\Delta_1 \cap \Delta_2$ is an exterior arc on Δ_1 and an interior arc on Δ_2.

Δ_1 intersects the yz plane in one arc, and Δ_2 intersects the plane in another arc, forming a t. Without loss of generality, let $\Delta_1 \cap \Delta_3$ be an exterior arc on Δ_3 and thus $\Delta_2 \cap \Delta_3$ must be an interior arc on Δ_3. $s_3 = \partial\Delta_3$ is a convex curve in the xz plane that contains the end points of $\Delta_3 \cap \Delta_1$ but does not intersect the arc $\Delta_3 \cap \Delta_2$ at all (since that is an interior arc on Δ_3, and thus is contained on the interior of Δ_3 missing its boundary). Up to isotopy, there is a unique way to choose a convex curve in the plane with this property. This means that there is a unique way up to isotopy to choose the link components, and, therefore,

Lemma 2.2. *The Borromean rings, are the unique link with three convex planar components, bounding planar disks, with one triple point in their intersection and one exterior arc and one interior arc on each of the three disks.*

This completes the proof on three components, because we have seen that if there are no triple points we have the unlink, if there is a triple point then we either have the Borromean rings (if each disk has one exterior arc and one interior arc) or a split link.

2.2. *The case of 4 convex components*

Let $L = s_1 \cup s_2 \cup s_3 \cup s_4$ be a link where each s_i is a convex planar curve bounding a planar disk Δ_i.

The first case is that there are *no triple points* of intersection for the disks. This means that there are at most six exterior arcs and six interior arcs on the disks (since there are $\binom{4}{2}$ ways to pair up these disks and form an intersection). Assume the disks are chosen with the smallest number of intersections possible. For all four disks to fail to have an outermost exterior arc on them, they must each have at least two interior arcs. Obviously this cannot be the case since there are a total of six interior arcs, not eight. Use one of the outermost exterior arcs to decrease the total number of intersections. This is a contradiction since we assumed we chose the disks to have a minimal number of intersections. Thus we must have no intersections, proving we have the unlink.

The second case is that there is *exactly one triple point* of intersection. This means that three of the disks, say Δ_1, Δ_2, and Δ_3 have a triple point on them, and the fourth Δ_4 does not. If $\Delta_1 \cap \Delta_4$ is an exterior arc on Δ_1 then it must be outermost on Δ_1 (the "t" must fall on one side of the arc or the other and that takes care of all of the other intersections) and we can get rid of the intersection. The same is, of course, true for Δ_2 and Δ_3. Thus, we can assume that all of the intersections are exterior arcs on Δ_4 since we can get rid of all interior arcs on it. This, however implies we have a split link by Lemma 2.1.

The third case is that we have *exactly two triple points.* Since a disk can only have 3 intersection arcs on it, we cannot see two "t's" on one disk. We must, therefore, on one of the disks, see one edge that has two other edges cross it forming an "H" intersection pattern. The edge that crosses the other two is part of two disks, so we must see an "H" on two disks and a "t" on two disks. Let Δ_1 and Δ_2 be the two disks containing the "$H's$". Call the arc that intersects two other arcs a_1 on Δ_1 and a_2 on Δ_2. Δ_3 and Δ_4, the two disks with the "t's" may intersect each other, but if so the exterior arc is outermost on the disk that contains it and so the intersection can be eliminated. Without loss of generality let a_1 be the exterior arc and a_2 be the interior arc.

Δ_1 must contain an interior arc or we have a split link, so we may assume that that arc is b_1, the intersection with Δ_3. If we have a triple point that corresponds to one exterior arc and one interior arc on each disk, by Lemma 2.2 the link contains the Borromean Rings as a sublink and thus is not Brunnian itself. Thus, since a_1 is an exterior arc and b_1 is an interior arc, a_2 and d_2 are both interior arcs and b_3 and d_3 are both exterior arcs.

We now notice that both of the arcs on Δ_3 are exterior arcs (and we already determined there are only two arcs of intersection on Δ_3 since $f = \Delta_3 \cap \Delta_4$ can be removed), so by Lemma 2.1, we have a split link!

The fourth case is that we have *exactly three triple points.* Note that each triple point appears on exactly three disks (the three that intersect to form it). If one disk has no triple points on it, then there can be only a total of at most one triple point. If one disk has exactly one triple point on it, then there can be at most two triple points total again showing we are actually in a previous case, not this one. Every disk, therefore, must have two or three triple points, since it is impossible for a disk to have more than three triple points when they are created by intersections of three (straight) arcs. Since each triple point appears on three disks, giving us a total of nine points to distribute on the disks, we must have one disk with three triple points and three disks with two triple points on them, as in Fig. 7. It turns out that without loss of generality, we can determine the exterior arcs and interior arcs on the disks to match those in the figure as follows:

1. Δ_1 must have an interior arc, so without loss of generality, we may assume a_1 is an interior arc and a_2 on Δ_2 is an exterior arc.

2. Not all arcs on Δ_2 can be exterior arcs, so we can assume d_2 is an interior arc on Δ_2 and d_3 is an exterior arc on Δ_3.
3. So that the triple point from a, b, d does not result in the Borromean rings, b_3 must be an exterior arc, and b_1 an interior arc.
4. Now e_3 must be a interior arc or all the arcs on Δ_3 are exterior arcs and thus e_4 is an exterior arc.
5. So that the triple point from e, b, c does not result in the Borromean rings, c_4 must be an exterior arc, and c_1 must be an interior arc.
6. Finally, f_4 must be an interior arc or D_4 has only exterior arcs.

This implies that Fig. 7 is the only possible case for three triple points. Note, though, that Δ_1 only contains interior arcs. This, however, is a contradiction since if we call the plane containing Δ_1, P, then $P - \Delta_1$ is disjoint from all of the disks. This, however, implies s_1 bounds a disk disjoint from the other knots and we have a split link (this is, of course, obvious in S^3 as the one point compactification of R^3, but the compactification, of course, has no affect on whether the link is split or not).

The final case is we have *4 triple points*. This means we have 3 triple points on each disk. No disk can have three exterior arcs, so the first case is we see three disks with two exterior arcs and one interior arc, and one disk with three interior arcs, but this implies a split link just as it did in the previous case. The other possible case is we have two disks with two exterior arcs and one interior arc, and two disks with two interior arcs and one exterior arc, such as Fig. 8. Let Δ_1 and Δ_2 be the former disks and Δ_3 and Δ_4 be the latter.

1. We may assume that a_1 is an interior arc since the interior arc on Δ_1 or Δ_2 had to result from their intersection with each other, thus a_2 is an exterior arc. This also mandates that b_1 and c_1 are exterior arcs and b_3 and c_4 are interior arcs since Δ_1 has only one interior arc on it.
2. This means that either d_2 or e_2 must be the interior arc on Δ_2 and without loss of generality we can assume that d_2 is an interior arc and e_2 an exterior arc (and thus d_3 is an exterior arc and e_4 an interior arc).
3. Since Δ_1 only had one interior arc, b_1 and c_1 are exterior arcs and thus b_3 and c_4 are interior arcs.
4. Finally, f_4 must be an exterior arc since Δ_4 contains an exterior arc and this is the only arc that is not labeled and thus f_3 is an interior arc. components

The a, b, d intersection, however, violates the triple point intersection rule implying that Δ_1, Δ_2, and Δ_3 form the Borromean rings, which shows this is not a Brunnian link.

Thus, no matter how many triple points we have there is a contradiction and we have not built a Brunnian link. □

3. Open Questions

We end with a few remaining open questions.

Question 3.1. Is there a Brunnian link other than the Borromean rings that can be formed out of convex curves?

The combinatorics get much more complicated as the number of components goes up. Davis [1] builds on this paper in his master's thesis at Wake Forest University to confirm that there are no convex Brunnian links for $n = 5$. It is possible (but certianly not obvious) that the same techniques may work for $n = 6$, but a new strategy seems necessary for large enough n.

Question 3.2. For a fixed $n \geq 3$ how many n component Brunnian links can be formed out of planar curves

As we observed in the introduction, for each n the answer is at least one, but perhaps all such examples are from the relatively small family of Brunnian links that are formed by iterated doubles of the Hopf link.

Question 3.3. Is there a Brunnian link other than the Borromean rings with a projection onto the plane such that the images of the components are all convex (creating an optical illusion such as the circular Borromean rings projection)?

Acknowledgment

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