AN INFINITE FAMILY OF CONVEX BRUNNIAN LINKS IN $\mathbb{R}^n$

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Abstract. This paper proves that convex Brunnian links exist for every dimension $n \geq 3$ by constructing explicit examples. These examples are three-component links which are higher-dimensional generalizations of the Borromean rings.

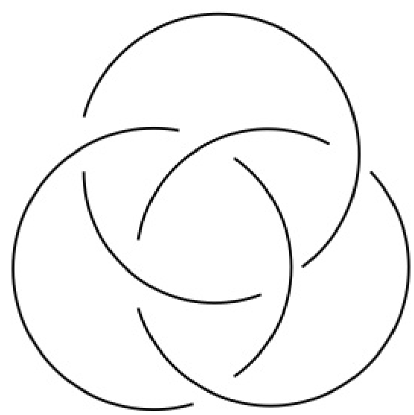

Figure 1. The Borromean Rings

1. Introduction

The link depicted in Figure 1 is known as the Borromean rings and appears to consist of three round circles. This, however, was proven to be an optical illusion by Mike Freedman and Richard Skora in [4], who showed that at least one component must be noncircular. A different proof of this result was given by Bernt Lindström and Hans-Olov Zetterström in [8]; it seems they were not aware of the earlier result.

Brunnian links were introduced over a hundred years ago by Hermann Brunn in his 1892 paper "Uber Verkettung" (*"On Linking"*) [1]. They have been generalized both in $\mathbb{R}^3$ as well as in higher dimensions. Debrunner [3] and Penney [9] each looked at generalizations of Brunnian links in $\mathbb{R}^3$ in 1961 and 1969 respectively. Takaaki Yanagawa was the first to look at higher-dimensional Brunnian links such as the ones we study in this paper back in 1964 when he constructed 2-spheres in $\mathbb{R}^4$ that formed Brunnian links [10].

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Instead of linked circles in $\mathbb{R}^3$, one can consider linked spheres of various dimensions in $\mathbb{R}^n$. In [7] it is shown that no Brunnian link in $\mathbb{R}^n$ can ever be built out of round spheres for any $n \in \mathbb{Z}^+$.

However, the Borromean rings can be built out of two circles and one ellipse that is arbitrarily close to a circle by using the equations

$$K_1 = \{x_1^2 + x_2^2 = r_1^2,\ x_3 = 0\}$$
$$K_2 = \{x_2^2 + x_3^2 = r_2^2,\ x_1 = 0\}$$
$$K_3 = \{\frac{x_1^2}{(r_3)^2} + \frac{x_3^2}{(r_4)^2} = 1,\ x_3 = 0\},$$

where $r_3 < r_1 < r_2 < r_4$, all arbitrarily close to each other.

In [6] it is proven that although there are an infinite number of Brunnian links of 3 components in $\mathbb{R}^3$ (or any number of components ≥ 3), the Borromean rings are the only Brunnian link in dimension three of either 3 or 4 components that can be built out of convex components. The result was extended to 5 components in [2].

The question of whether any Brunnian link in $\mathbb{R}^n$ ($n \geq 3$) other than the Borromean rings can be built out of convex components was asked in [7, 2]. The main result of this answers that question in the affirmative:

Theorem 3.1. *Consider the infinite family $L_{i,n}$ of three-component links given explicitly by* (1)*; each consists of a round* (n–2)*-sphere, a round* (n–i–1)*-sphere and an i-dimensional ellipsoid sitting in $\mathbb{R}^n$ (for $1 \leq i \leq n$–2 and $n \geq 3$). Each $L_{i,n}$ is a convex Brunnian link.*

As we will see, each of these links is a natural generalization of the Borromean rings. Moreover, each can be constructed so the ellipsoid is arbitrarily close to being round.

This is organized as follows: the next section covers the background and relevant definitions for higher-dimensional linking. The main portion of this is section 3, which proves Theorem 3.1.

In section 4, we provide a second proof of Theorem 3.1 for the special case $i = 1$, in which we explicitly realize the first component as an ellipse (an S^1) and the other two as round (n–2)-spheres. This second proof uses the fundamental group as its main tool but does not extend to the general case. Section 5 concludes this with questions and conjectures about other convex Brunnian links: do they exist? Do the Borromean rings generalize to three (n–2)-spheres sitting in $\mathbb{R}^n$?

2. Standard definitions

Recall that a *knot* is a subset of $\mathbb{R}^3$ or S^3 that is homeomorphic to a circle (also called a 1-sphere or S^1). If the knot bounds an embedded disk it is called an *unknot*; otherwise it is knotted.

A *link* L is a collection of disjoint knots. A link L is an *unlink* of n components if it consists of n unknots and if the components simultaneously bound disjoint embedded disks.

A *Brunnian link* L is a link of $n \geq 3$ components that is not an unlink, but every proper sublink of L is an unlink. The *Borromean rings* form the most famous example of a Brunnian link (Figure 1). Note that eliminating any one of the components yields an unlink.

A subset K of $\mathbb{R}^n$ is a *knot in* $\mathbb{R}^n$ if K is homeomorphic to S^k for some k. By a *link in* $\mathbb{R}^n$ is meant a subset L of $\mathbb{R}^n$ that is homeomorphic to a disjoint union of finitely many knots (possibly of different dimensions). Knot theory is usually restricted to the case where $n = 3$; each knot is homeomorphic to S^1.

A link $L = F_1 \cup F_2 \cup \dots F_m$ in $\mathbb{R}^n$ is *an unlink in* $\mathbb{R}^n$ if for each i the knot F_i bounds a ball B_i (of appropriate dimension) such that $B_i \cap F_j = \emptyset$ $(i \neq j)$. If $n = 3$ and we restrict to circles, we obtain the traditional definition of an unlink, where each component bounds a disk disjoint from the other components. If a link L (in $\mathbb{R}^n$) of $m(\geq 3)$ components is not an unlink (in $\mathbb{R}^n$), yet every proper sublink is an unlink (in $\mathbb{R}^n$), we call L a *Brunnian link in* $\mathbb{R}^n$.

We say a link L is a *split link in* $\mathbb{R}^n$ if there is an $(n-1)$-sphere that is disjoint from the link and separates $\mathbb{R}^n$ into two components, each containing at least one component of the link. (The $(n-1)$-sphere need not be round.)

A knot S^k is said to be *convex* if it bounds a ball B^{k+1} which is convex.

A *generalized Hopf link* is any link of two components, one an S^j and the other an $S^{n-(j+1)}$ in $\mathbb{R}^n$ or S^n, each of which bounds a ball that intersects the other sphere transversally in exactly one point. Examples include any link isotopic to the following link in $\mathbb{R}^n$: $F_1 = \{(x_1, x_2, \dots x_n) : x_{i+1}^2 + x_{i+2}^2 + \dots + x_n^2 = 1, x_1 = x_2 = \dots = x_i = 0\}$ and $F_2 = \{(x_1, x_2, \dots x_n) : x_1^2 + x_2^2 + \dots + x_i^2 + (x_n - 1)^2 = 1, x_{i+1} = x_{i+2} = \dots = x_{n-1} = 0\}$, where $1 \leq i \leq n-2$.

3. An infinite family of convex Brunnian links

In this section, we present our main result, the existence of an infinite family of three-component convex Brunnian links in $\mathbb{R}^n$. Define the family $L_{i,n}$ as

$$L_{i,n} = K_1 \cup K_2 \cup K_3 \tag{1}$$

$$K_1 = \{(x_1, x_2, \dots x_n) : x_1^2 + x_2^2 + \dots + x_{n-1}^2 = 4,\ x_n = 0\}$$
$$K_2 = \{(x_1, x_2, \dots x_n) : x_{i+1}^2 + x_{i+2}^2 + \dots + x_n^2 = 9,\ x_1 = x_2 = \dots = x_i = 0\}$$
$$K_3 = \{(x_1, x_2, \dots x_n) : x_1^2 + x_2^2 + \dots + x_i^2 + \tfrac{x_n^2}{16} = 1, x_{i+1} = \dots = x_{n-1} = 0\},$$

where $n \geq 3$ and $1 \leq i \leq n-2$.

Theorem 3.1. *Each $L_{i,n}$ is a convex Brunnian link.*

We know that $L_{1,3}$ consists of two circles and an ellipse forming the Borromean rings, a Brunnian link. In the general case it is clear that the components are convex. The following lemma shows that all sublinks of $L_{i,n}$ are unlinks. We must show that no $L_{i,n}$ is an unlink, which we accomplish via Lemmas 3.3-3.5.

Henceforth, we adopt the convention for specifying knots and balls that all omitted coordinates are set equal to be zero.

Lemma 3.2. *Every proper sublink of $L_{i,n}$ is an unlink.*

Proof. The proper sublinks of $L_{i,n}$ are pairs of unknots. For each pair, we observe that one knot bounds a ball disjoint from the other, and thus the pair forms a split link and must be an unlink.

Explicitly, the round (n–1)-ball $B_1 = \{x_1^2+x_2^2+\cdots+x_{n-1}^2 \leq 4\} \subset \mathbb{R}^n$ bounded by K_1 lies in the complement of the (n–i–1)-sphere K_2. Similarly, K_2 bounds a (round) ball B_2 disjoint from the i-dimensional ellipsoid K_3, and K_3 bounds an ellipsoidal ball B_3 disjoint from K_1. □

The next lemma, stated without proof, relays a standard fact about spheres.

Lemma 3.3. *Let $S^n \subset \mathbb{R}^{n+1}$ be a round sphere centered at the origin, and let V be a linear subspace of $\mathbb{R}^{n+1}$. Consider the great spheres $X_1 = V \cap S^n$ and $X_2 = V^{\perp} \cap S^n$. Then, $S^n - X_1$ deformation retracts onto X_2.*

In particular, if we delete an unknotted $S^{k-1} \subset S^n$ from S^n, the result deformation retracts onto an S^{n-k}.

Next, we note that $K_3 - (B_1 \cap K_3)$ is not connected, since the intersection $(B_1 \cap K_3)$ is exactly the equator $x_n = 0$ of the i-dimensional ellipsoid K_3. The two components of $K_3 - (B_1 \cap K_3)$ are the upper and lower open halves of K_3. We use the upper half for our next definition.

Let K_3' be the i-dimensional subset of $K_3 \cup B_1$ formed by taking the open upper half of the ellipsoid K_3 and closing it at the bottom with the disk $B_1 \cap B_3$; in coordinates,

$$\begin{aligned} K_3' = &\left\{(x_1, x_2, \ldots x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \tfrac{x_n^2}{16} = 1, x_n \geq 0\right\} \\ &\cup \left\{(x_1, x_2, \ldots x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 \leq 1, x_n = 0\right\}. \end{aligned}$$

Lemma 3.4. *$K_3' \cup K_2$ is a generalized Hopf link and is not a split link.*

Proof. Figure 2 depicts this link for $n = 3$; notice that K_3' orthogonally intersects B_2 at the origin, so this a Hopf link. In arbitrary dimensions, the same phenomenon occurs: K_3' orthogonally intersects B_2 in only one point, the origin, and we have a generalized Hopf link.

(By letting B_3' be the portion of B_3 with $x_n \geq 0$, i.e., the portion bounded by K_3', we also see that K_2 orthogonally intersects B_3' in only one point, $q_+ = (0, 0, \ldots, 0, 0, 3)$.)

Now we show the link is not split. Although $K_3' \cup K_2$ lies in $\mathbb{R}^n$, our argument is more easily made in S^n. If we include $\mathbb{R}^n$ inside $\mathbb{R}^{n+1}$ by fixing $x_{n+1} = 0$, then we

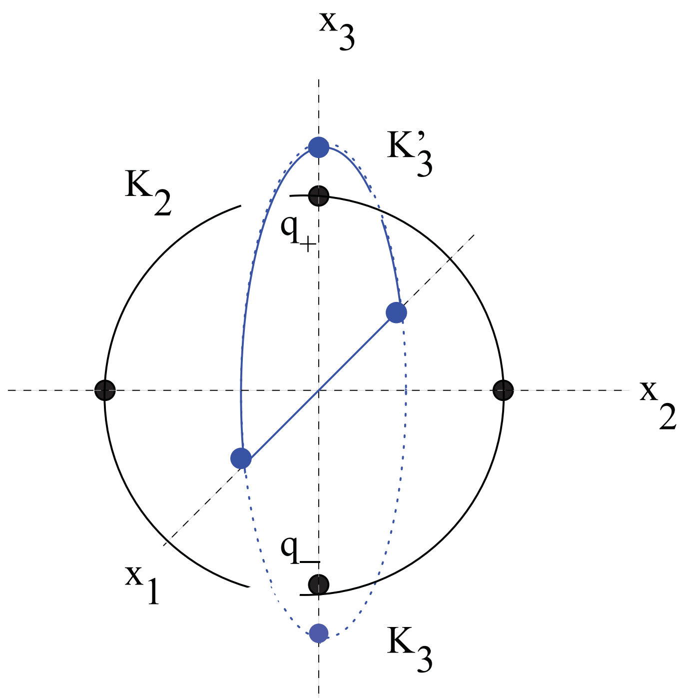

FIGURE 2. Here is how the argument looks for $L_{1,3} \subset \mathbb{R}^3$. Ball B_2 has dimension $n-i=2$; it forms the round disk bounded by K_2 in the x_2x_3-plane; the $(i+1=2)$-ball B_3' is the upper half-disk bounded by K_3' in the x_1x_3-plane. Let p be the map via stereographic projection from $\mathbb{R}^3$ to $S^3 \subset \mathbb{R}^4$. Then we see that $p(K_2)=G_2$, $p(K_3')=G_3'$, $p^{-1}(G_3)$ is the x_1-axis, and $p^{-1}(\Sigma)$ is the x_1x_3-plane. Note that q_+ is 'inside' K_3' in the x_1x_3-plane, whereas q_- is 'outside'.

may use stereographic projection p to lift $\mathbb{R}^n$ to the n-sphere $S^n \subset \mathbb{R}^{n+1}$ of radius 3, centered at the origin. (Usually the unit sphere is used for stereographic projection, but in this case it is more convenient to use the radius of K_2.) We will show the lift of link $K_3' \cup K_2$ is not split.

Because K_2 has radius 3 and is centered at the origin, it is fixed by the lift p. We consider these subsets of S^n:

- G_2, the great $(n$–i–$1)$-sphere $p(K_2)$
- G_3, the great i-sphere $\left\{x_1^2+x_2^2+\cdots+x_i^2+x_{n+1}^2=9\right\}$ in S^n complementary to G_2
- G_3', the lift $p(K_3')$, and
- Σ, the great $(i+1)$-sphere $\left\{x_1^2+x_2^2+\cdots+x_i^2+x_n^2+x_{n+1}^2=9\right\}$

We note that both G_3 and G_3' are unknotted i-spheres contained in $\Sigma \subset S^n$; the former inclusion is immediate while the latter follows since K_3' lies in the $(i+1)$-dimensional subspace $p^{-1}(\Sigma)$. We observe that $G_2 \cap \Sigma$ is a great 0-sphere consisting of two points $\tilde{q}_\pm = (0,0,\dots,0,\pm 3,0)$ which are disjoint from $G_3 \cup G_3'$. Note that $\tilde{q}_\pm = p(q_\pm)$.

Claim: G_3' is isotopic to G_3 in $\Sigma - G_2$ (which implies they are isotopic in $S^n - G_2$).

To prove the claim, it suffices to show both G_3 and G_3' separate $\tilde{q}_+$ from $\tilde{q}_-$ in Σ, i.e, they can both be oriented to contain $\tilde{q}_+$ on the inside and $\tilde{q}_-$ on the outside. This follows immediately for G_3 since it is a great i-sphere disjoint from antipodal points $\tilde{q}_\pm$ in Σ.

Now we show G_3' separates $\tilde{q}_\pm$. Recall that K_3' bounds B_3', which contains q_+ but not q_-. Also note that $B_3' \subset p^{-1}(\Sigma)$. This property is preserved under homeomorphism p since p is a bijection and $B_3' \subset p^{-1}(\Sigma)$ and $p(B_3') \subset \Sigma$. The i-sphere G_3' bounds $p(B_3')$, which contains $\tilde{q}_+$ but not $\tilde{q}_-$. Thus G_3' separates $\tilde{q}_\pm$ and the claim holds. Since G_3 and G_3' are isotopic in $\Sigma - G_2$, they are also isotopic in the larger space $S^n - G_2$ which contains $\Sigma - G_2$.

We have now shown that $K_2 \cup K_3'$ lifts to a link isotopic to $G_2 \cup G_3$. To prove the former is not split, we show the latter is not split by showing that G_3 does not bound an $(i+1)$-ball in the complement of G_2. Lemma 3.3 assures us that $S^n - G_2$ deformation retracts onto G_3; this implies that $\pi_i(S^n - G_2) = \pi_i(G_3) = \mathbb{Z}$. (It is well known that the ith homotopy group of an i-sphere is $\mathbb{Z}$; see, for example, [5]). Since G_3 is fixed by the deformation retract and generates π_i after the deformation retract, it must also generate π_i before. As a nontrivial element of $\pi_i(S^n - G_2)$, G_3 cannot bound a $(i+1)$-ball in $S^n - G_2$. Therefore, $G_2 \cup G_3 \subset S^n$ is not a split link, and neither is $K_2 \cap K_3' \subset \mathbb{R}^n$. □

Lemma 3.5. *If K_2 bounds an embedded $(n-i)$-ball D_2 which does not intersect $K_1 \cup K_3$, then K_2 bounds an immersed $(n-i)$-ball D_2' that does not intersect $K_1 \cup K_3 \cup B_1$.*

Proof. Let D_2 be an embedded $(n-i)$-ball bounded by K_2 that is disjoint from $K_1 \cup K_3$ and intersects B_1 transversally. We want to show there is an immersed $(n-i)$-ball that is disjoint from B_1. If D_2 is disjoint from B_1, we are done. If not, note that B_1 has dimension $n-1$ and thus has codimension 1 in $\mathbb{R}^n$. Since $\partial B_1 \cap D_2 = \emptyset$ and $\partial D_2 \cap B_1 = \emptyset$, the set $B_1 \cap D_2$ must be a collection of disjoint closed manifolds $\{F_1, F_2, \dots\}$ of dimension $n-i-1$.

We note that since D_2 is a ball, each of the F_i are separating in D_2. We may define the *outside* of F_i to be the component of $D_2 - F_i$ which contains K_2 and the *inside* to be the other component.

Since D_2 is compact, we may assume that it has a finite number of critical points with respect to x_n. Since B_1 lies in the plane $x_n = 0$, we can conclude that the intersection $B_1 \cap D_2$ has a finite number of components. Because there are a finite number of intersections, we may take an innermost component F_j (one which has no other F_i inside of it in D_2). Let U be the component of $D_2 - F_j$ inside of F_j.

Let $f(x_1, x_2, \dots x_{n-1}, x_n) = (x_1, x_2, \dots x_{n-1}, |x_n|)$. Note that p is a point in K_i if and only if $f(p)$ is a point in K_i since each of the knots is symmetric with respect to x_n.

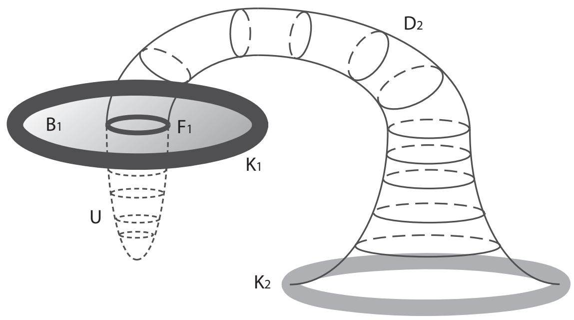

FIGURE 3. We may cut out U and paste in $f(U)$ to eliminate the intersections of B_1 and D_2. In this figure, set in $\mathbb{R}^3$, we have simplified the link by omitting K_3.

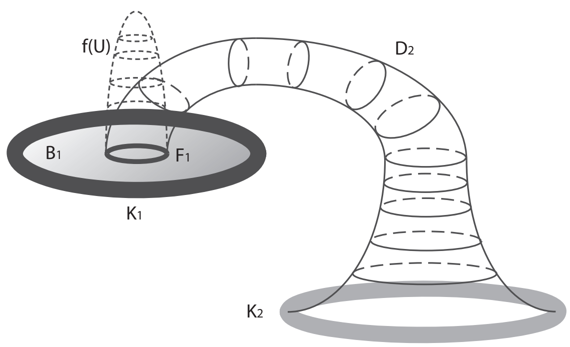

FIGURE 4. We reflect $U \subset D_2$ across B_1 to reduce the number of intersections.

This implies that if $U \subset D_2$ then since $U \cap (K_1 \cup K_3) = \emptyset$, we know $f(U) \cap (K_1 \cup K_3) = \emptyset$.

Then we replace D_2 by $(D_2 - U) \cup f(U)$. See Figures 3-4. The new ball is not in general position with respect to B_1. We may take a small deformation of the new (possibly no longer embedded) ball to decrease the number of intersections with B_1. We repeat, reducing the number of intersections each time, until we have a new ball D_2' with boundary K_2 that is disjoint from B_1 and $K_1 \cup K_3$. $\square$

Proof of Theorem 3.1. We now complete the proof of our main theorem. Observe that if $L_{i,n}$ is an unlink, then K_2 bounds a ball D_2 disjoint from $K_1 \cup K_3$, but Lemma 3.5 shows that this implies K_2 also bounds an immersed ball D_2' that does not intersect $K_1 \cup K_3 \cup B_1$. The existence of D_2' implies that K_2 is homotopically trivial in the complement of $K_1 \cup K_3 \cup B_1$. Since $K_3' \subset K_3 \cup B_1$, we know K_2 must also be homotopically trivial in the complement of K_3'. This, however, contradicts Lemma 3.4. $\square$

We note that by replacing our coefficients $(1, 4, 9,$ and $16)$ in the $L_{i,n}$ formulas by coefficients that are arbitrarily close to each one, say $(1, 1+\epsilon, 1+2\epsilon,$ and $1+3\epsilon)$, we obtain embeddings of the same links containing two round spheres and a third that is arbitrarily close to being a round sphere. Thus, although [7] shows that no Brunnian link can be built out of round components, this paper provides an infinite collection of Brunnian links in which two of the components are round and the third is arbitrarily close to being round.

4. A special case

We have now proven our theorem, but a different proof technique exists for a subset of the links. For the links $L_{1,n}$, one of the components of the link is homeomorphic to a circle, and we can utilize the fundamental group instead for our proof. Here we allow K_3 to be the (elliptical) circle, but symmetry dictates that the proof holds if any of the other components had been the circle.

Theorem 4.1. *$L_{1,n}$ is a convex Brunnian link.*

Proof. We want to show that the loop K_3 does not bound a disk disjoint from $K_1 \cup K_2$. We do so by showing it is nontrivial in the fundamental group of $\mathbb{R}^n - (K_1 \cup K_2)$. We first prove the following lemma.

Lemma 4.2. *$\pi_1(\mathbb{R}^n - (K_1 \cup K_2))$ is the free group on two generators.*

This follows from the following well known lemma.

Lemma 4.3. *The groups $\pi_1(S^n - J_i)$, $\pi_1(\mathbb{R}^n - K_i)$, and $\pi_1(B^n - K_i)$ are all isomorphic to $\mathbb{Z}$ if K_i is a round $(n-2)$-sphere in $B_n \subset \mathbb{R}^n$ and J_i is a round $(n-2)$-sphere in S^n.*

Proof of Lemma 4.3. Both the complement of B^n and the boundary of B^n are simply connected in $\mathbb{R}^n$ (and the same is true of a round ball in S^n) and thus the Seifert-Van Kampen Theorem shows that the fundamental group of the larger space is only dependent on the fundamental group of the ball. Now we know that $\pi_1(S^n - J_i)$, $\pi_1(\mathbb{R}^n - K_i)$, and $\pi_1(B^n - K_i)$ are all isomorphic to each other and we need only show that $\pi_1(S^n - J_i)$ is isomorphic to $\mathbb{Z}$. This follows from Lemma 3.3, which states that if we take a round sphere $S^{n-2} \subset S^n$, then $S^n - S^{n-2}$ deformation retracts onto S^1 and therefore has the fundamental group of a circle. $\square$

Proof of Lemma 4.2. Since K_1 and K_2 are unlinked, there exists an embedded S^{n-1} that separates them. Therefore the Seifert-Van Kampen Theorem implies that the fundamental group of $\mathbb{R}^n - (K_1 \cup K_2)$ is the free product of the fundamental groups of $B^n - K_1$ and $B^n - K_2$, i.e., the free group on two generators. $\square$

Now to conclude the proof of Theorem 4.1, we observe that in the case of $L_{1,n}$, the fundamental group of $\mathbb{R}^n - (K_1 \cup K_2)$ has generators

$$\begin{aligned} \alpha(t) &= (\;3\sin 2\pi t,\; 0,\, 0, \ldots 0,\; 3 - 3\cos 2\pi t) \\ \beta(t) &= (2 - 2\cos 2\pi t,\; 0,\, 0, \ldots 0,\; 2\sin 2\pi t\;) \end{aligned} \qquad (0 \le t \le 1),$$

with the origin as the base point. Let $\gamma(t) = (t, 0, 0, \ldots, 0)$. We orient K_3 starting from the point $(1, 0, \ldots, 0)$ and move initially in the positive x_n direction.

We observe that $\gamma K_3 \gamma^{-1}$ is homotopic to $\alpha\beta^{-1}\alpha^{-1}\beta$, which is a commutator of the two generators. (The circle α is clearly homotopic to the loop made by following the top half of K_3 then closing it by following the x_1-axis. The hard part of this observation is noticing that $\beta^{-1}\alpha^{-1}\beta$ is homotopic to a curve that follows the bottom half of ellipse K_3 and then returns along the x_1-axis.) Since the commutator is nontrivial, K_3 must represent a nontrivial element in $\pi_1(\mathbb{R}^n - (K_1 \cup K_2))$. Thus, $L_{1,n}$ is not an unlink but its proper sublinks are unlinks; hence is $L_{1,n}$ is Brunnian. □

5. Open Questions

Here we list a few conjectures regarding the following infinite family of convex links.

$$L_{i,j,n} = K_1 \cup K_2 \cup K_3 \tag{2}$$

$$\begin{aligned}
K_1 &= \{(x_1, x_2, \ldots x_n) : x_1^2 + x_2^2 + \cdots + x_{n-j-1}^2 = 4,\ \ x_{n-j} = \cdots = x_n = 0\} \\
K_2 &= \{(x_1, x_2, \ldots x_n) : x_{i+1}^2 + x_{i+2}^2 + \cdots + x_n^2 = 9,\ \ x_1 = x_2 = \cdots = x_i = 0\} \\
K_3 &= \{(x_1, x_2, \ldots x_n) : x_1^2 + x_2^2 + \cdots + x_i^2 + \frac{x_{n-j}^2 + \cdots + x_{n-1}^2 + x_n^2}{16} = 1, \\
&\qquad\qquad x_{i+1} = x_{i+2} = \cdots = x_{n-j-1} = 0\}
\end{aligned}$$

These links generalize the family $L_{i,n}$ considered in section 3, in that $L_{i,0,n} = L_{i,n}$.

Conjecture 5.1. All the links in the family $L_{i,j,n}$ are Brunnian.

Conjecture 5.2. There exist convex Brunnian links that are not isotopic to a link of the form $L_{i,j,n}$.

No such links exist in $\mathbb{R}^3$ for 3, 4, and 5 component links by [6, 2]. We speculate that adding more components in $\mathbb{R}^3$ will not produce examples. However, in higher dimensions it seems likely that other convex Brunnian links exist. In particular, is there such a link comprised of three $(n-2)$-dimensional knots for $n > 3$? Aside from the Borromean rings in $\mathbb{R}^3$, this case is impossible within our families (1) and (2).

Conjecture 5.3. Although no Brunnian link can be built out of round spheres (see [7]), it is true that for any $\epsilon > 0$, our families (1) and (2) of links may be isotoped so that K_1 and K_2 remain as round spheres and K_3 is contained in an ϵ-neighborhood of a round sphere. We conjecture that all convex Brunnian links can be made arbitrarily close to perfectly round in this manner.

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