EVERY GRAPH HAS AN EMBEDDING IN S^3 CONTAINING NO NON-HYPERBOLIC KNOT

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In contrast with knots, whose properties depend only on their extrinsic topology in S^3, there is a rich interplay between the intrinsic structure of a graph and the extrinsic topology of all embeddings of the graph in S^3. For example, it was shown in [2] that every embedding of the complete graph K_7 in S^3 contains a non-trivial knot. Later in [3] it was shown that for every $m \in \mathbb{N}$, there is a complete graph K_n such that every embedding of K_n in S^3 contains a knot Q (i.e., Q is a subgraph of K_n) such that $|a_2(Q)| \geq m$, where a_2 is the second coefficient of the Conway polynomial of Q. More recently, in [4] it was shown that for every $m \in \mathbb{N}$, there is a complete graph K_n such that every embedding of K_n in S^3 contains a knot Q whose minimal crossing number is at least m. Thus there are arbitrarily complicated knots (as measured by a_2 and the minimal crossing number) in every embedding of a sufficiently large complete graph in S^3.

In light of these results, it is natural to ask whether there is a graph such that every embedding of that graph in S^3 contains a composite knot. Or more generally, is there a graph such that every embedding of the graph in S^3 contains a satellite knot? Certainly, K_7 is not an example of such a graph since Conway and Gordon [2] exhibit an embedding of K_7 containing only the trefoil knot. In this paper we answer this question in the negative. In particular, we prove that every graph has an embedding in S^3 such that every non-trivial knot in that embedding is hyperbolic. Our theorem implies that every graph has an embedding in S^3 which contains no composite or satellite knots. By contrast, for any particular embedding of a graph we can add local knots within every edge to get an embedding such that every knot in that embedding is composite.

Let G be a graph. There is an odd number n, such that G is a minor of K_n. We will show that for every odd number n, there is an embedding of K_n in S^3 such that every non-trivial knot in that embedding of K_n is hyperbolic. It follows that there is an embedding of G in S^3 which contains no non-trivial non-hyperbolic knots.

Let n be a fixed odd number. We begin by constructing a preliminary embedding of K_n in S^3 as follows. Let h be a rotation of S^3 of order n with fixed point set $\alpha \cong S^1$. Let V denote the complement of an open regular neighborhood of the fixed point set α. Let $v_1, \ldots, v_n$ be points in V such that for each i, $h(v_i) = v_{i+1}$ (throughout the paper we shall consider

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our subscripts mod n). These v_i will be the vertices of the preliminary embedding of K_n.

Definition 1. *By a* **solid annulus** *we shall mean a 3-manifold with boundary which can be parametrized as $D \times I$ where D is a disk. We use the term* **the annulus boundary** *of a solid annulus $D \times I$ to refer to the annulus $\partial D \times I$. The* **ends** *of $D \times I$ are the disks $D \times \{0\}$ and $D \times \{1\}$. If A is an arc in a solid annulus W with one endpoint in each end of W, and A co-bounds a disk in W together with an arc in ∂W, then we say that A is a* **longitudinal arc** *of W.*

As follows, we embed the edges of K_n as simple closed curves in the quotient space $S^3/h = S^3$. Observe that since V is a solid torus, $V' = V/h$ is also a solid torus. Let D' denote a meridional disk for V' which does not contain the point $v = v_1/h$. Let W' denote the solid annulus $\mathrm{cl}(V' - D')$ with ends D'_+ and D'_-. Since n is odd, we can choose unknotted simple closed curves $S_1, \ldots, S_{\frac{n-1}{2}}$ in the solid torus V' such that each S_i contains v and has winding number $n + i$ in V', the S_i are pairwise disjoint except at v, and for each i, $W' \cap S_i$ is a collection of $n + i$ untangled longitudinal arcs (see Figure 1).

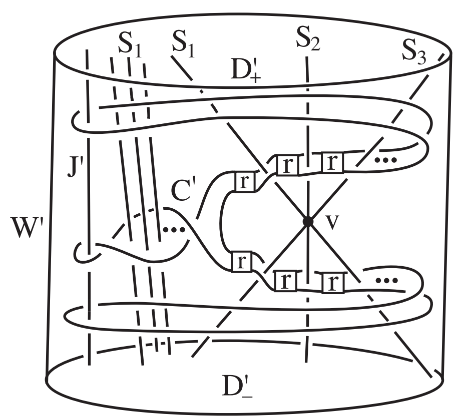

FIGURE 1. For each i, $W' \cap S_i$ is a collection of $n+i$ untangled longitudinal arcs.

We define two additional simple closed curves J' and C' in V' whose intersections with W' are illustrated in Figure 1 as follows. First, choose a simple closed curve J' in V', whose intersection with W' is a longitudinal arc which is disjoint from and untangled with $S_1 \cup \cdots \cup S_{\frac{n-1}{2}}$. Next we let C' be the unknotted simple closed curve in $W' - (S_1 \cup \cdots \cup S_{\frac{n-1}{2}} \cup J')$ whose projection is illustrated in Figure 1. In particular, C contains one half twist between J' and the set of arcs of $S_1 \cup \cdots \cup S_{\frac{n-1}{2}}$ which do not contain v, another half twist between those arcs of $S_1 \cup \cdots \cup S_{\frac{n-1}{2}}$ and the set of arcs containing v, and r full-twists between each of the individual arcs of S_i and S_{i+1} containing v. We will determine the value of r later.

Each of the $\frac{n-1}{2}$ simple closed curves $S_1, \ldots, S_{\frac{n-1}{2}}$ lifts to a simple closed curve consisting of n consecutive edges of K_n. The vertices $v_1, \ldots, v_n$ together with these $\frac{n(n-1)}{2}$ edges gives us a preliminary embedding Γ_1 of K_n in S^3.

Lift the meridional disk D' of the solid torus V' to n disjoint meridional disks $D_1, \ldots, D_n$ of the solid torus V. Lift the simple closed curve C' to n disjoint simple closed curves $C_1, \ldots, C_n$, and lift the simple closed curve J' to n consecutive arcs $J_1, \ldots, J_n$ whose union is a simple closed curve J. The closures of the components of $V - (D_1 \cup \cdots \cup D_n)$ are solid annuli, which we denote by $W_1, \ldots, W_n$. The subscripts of all of the lifts are chosen consistently so that for each i, $v_i \in W_i$, $C_i \cup J_i \subseteq W_i$, and D_i and D_{i+1} are the ends of the solid annulus W_i. For each i, the pair $(W_i - (C_i \cup J_i), (W_i - (C_i \cup J_i)) \cap \Gamma_1))$ is homeomorphic to $(W' - (C' \cup J'), (W' - (C' \cup J')) \cap (S_1 \cup \cdots \cup S_{\frac{n-1}{2}}))$. For each i, the solid annulus W' contains $n + i - 1$ arcs of S_i which are disjoint from v. Hence each edge of the embedded graph Γ_1 meets each solid annulus W_i in at least one arc not containing v_i.

Let κ be a simple closed curve in Γ_1. For each i, we let k_i denote the set of those arcs of $\kappa \cap W_i$ which do not contain v_i, and let e_i denote either the single arc of $\kappa \cap W_i$ which does contain v_i or the empty set if v_i is not on κ. Observe that since κ is a simple closed curve, it contains at least three edges of Γ_1; and as we observed above, each edge of κ contains at least one arc of k_i. Thus for each i, k_i contains at least three arcs. Either e_i is empty, the endpoints of e_i are in the same end of the solid annulus W_i, or the endpoints of e_i are in different ends of W_i. We illustrate these three possibilities for $(W_i, C_i \cup J_i \cup k_i \cup e_i)$ In Figure 2 as forms a), b) and c) respectively. The number of full-twists represented by the labels t, u, x, or z in Figure 2 is some multiple of r depending on the particular simple closed curve κ.

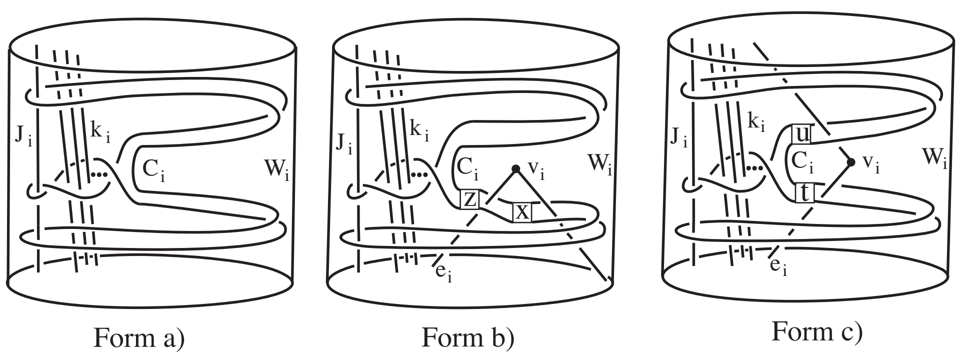

FIGURE 2. The forms of $(W_i, C_i \cup J_i \cup k_i \cup e_i)$.

For each of the forms of $(W_i, C_i \cup J_i \cup k_i \cup e_i)$ illustrated in Figure 2, we will associate an additional arc and an additional collection of simple closed curves as follows (illustrated in Figure 3). Let the arc B_i be the core of a solid annulus neighborhood of the union of the arcs k_i in W_i such that B_i is disjoint from J_i, C_i, and e_i. Let the simple closed curve Q be obtained from C_i by removing the full twists z, x, t, and u. Let Z, X, T, and U be unknotted simple closed curves which wrap around Q in place of z, x, t, and u as illustrated in Figure 3.

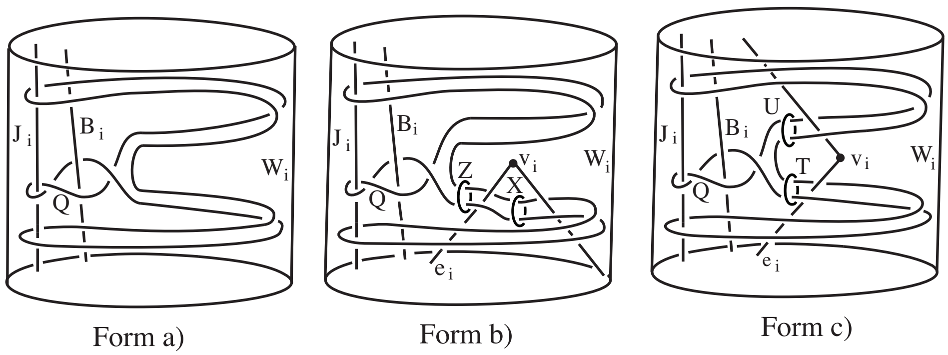

FIGURE 3. The forms of W_i with associated simple closed curves and the arc B_i.

For each i, let M_i denote an unknotted solid torus in S^3 obtained by gluing together two identical copies of W_i along D_i and D_{i+1}, making sure that the end points of the arcs of J_i, B_i, and e_i match up with their counterparts in the second copy to get simple closed curves j, b, and E respectively in M_i. Thus M_i has a 180° rotational symmetry around a horizontal line which goes through the center of the figure and the end points of both copies of J_i, B_i, and e_i. Recall that in form a), e_i is the empty set, and hence so is E. Let Q_1 and Q_2, X_1 and X_2, Z_1 and Z_2, T_1 and T_2, and U_1 and U_2 denote the doubles of the unknotted simple closed curves Q, X, Z, T, U respectively.

Let Y denote the core of the solid torus $\mathrm{cl}(S^3 - M_i)$. We associate to Form a) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y$. We associate to Form b) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup X_1 \cup X_2 \cup Z_1 \cup Z_2$. We associate to Form c) of Figure 3 the link $L = Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E \cup T_1 \cup T_2 \cup U_1 \cup U_2$. Figure 4 illustrates the three forms of the link L.

The software program SnapPea (available at http://www.geometrygames.org/SnapPea/index.html) can be used to determine whether or not a given knot or link in S^3 is hyperbolic, and if so SnapPea estimates the hyperbolic volume of the complement. We used SnapPea to verify that each of the three forms of the link L illustrated in Figure 4 is hyperbolic.

A 3-manifold is unchanged by doing Dehn surgery on an unknot if the boundary slope of the surgery is the reciprocal of an integer (though such surgery may change a knot or link in the manifold). According to Thurston's

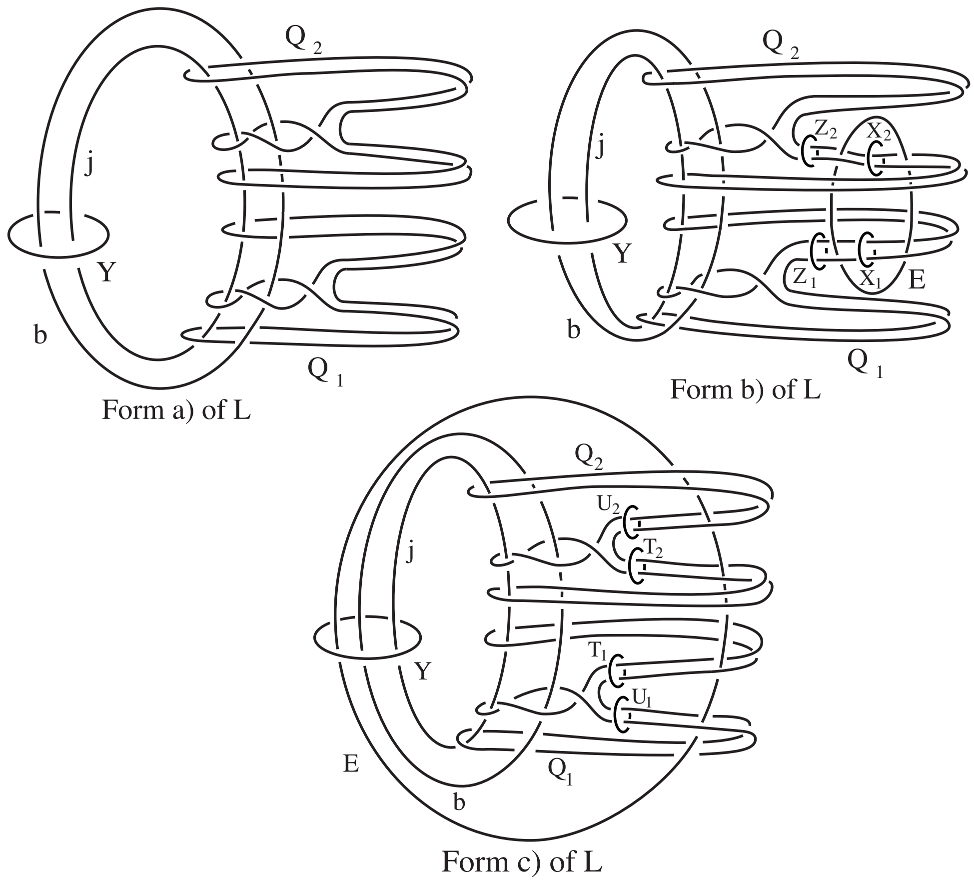

FIGURE 4. The possible forms of the link L.

Hyperbolic Dehn Surgery Theorem [1, 5], all but finitely many Dehn fillings of a hyperbolic link complement yield a hyperbolic manifold. Thus there is some $r \in \mathbb{N}$ such that for any $m \geq r$, if we do Dehn filling with slope $\frac{1}{m}$ along the components X_1, X_2, Z_1, Z_2 of the link L in form b) or along the components T_1, T_2, U_1, U_2 of the link L in form c), then we obtain a hyperbolic link $\overline{Q_1} \cup \overline{Q_2} \cup j \cup b \cup Y \cup E$, where the simple closed curves $\overline{Q_1}$ and $\overline{Q_2}$ are obtained by adding m full twists to Q_1 and Q_2 in place of each of the surgered curves.

We fix the value of r according to the above paragraph, and this is the value of r that we use in Figure 1. Recall that the number of twists x, z, u, and t in the simple closed curves C_i in Figure 2 are each a multiple of r. Thus the particular simple closed curves C_i are determined by our choice of r together with our choice of the simple closed curve κ. Now we do Dehn fillings along X_1 and X_2 with slope $\frac{1}{x}$, along Z_1 and Z_2 with slope $\frac{1}{z}$, along U_1 and U_2 with slope $\frac{1}{u}$, and along T_1 and T_2 with slope $\frac{1}{t}$. Since x, z, u, and t are each greater than or equal to r, the link $\overline{Q_1} \cup \overline{Q_2} \cup j \cup b \cup Y \cup E$

that we obtain will be hyperbolic. In form a), E is the empty set and the link $Q_1 \cup Q_2 \cup j \cup b \cup Y \cup E$ was already seen to be hyperbolic using SnapPea. In this case, we do no surgery and we let the simple closed curves $\overline{Q}_1 = Q_1$ and $\overline{Q}_2 = Q_2$. It follows that each form of $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is a hyperbolic 3-manifold. Observe that $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is the double of $W_i - (C_i \cup J_i \cup B_i \cup e_i)$.

Now that we have fixed C_i, we let $N(C_i)$, $N(J_i)$, $N(B_i)$, and $N(e_i)$ be pairwise disjoint regular neighborhoods of C_i, J_i, B_i, and e_i respectively in the interior of each of the forms of the solid annulus W_i (illustrated in Figure 2). We choose $N(B_i)$ such that it contains the union of the arcs k_i. Note that in form a) e_i is the empty set and hence so is $N(e_i)$. Let $N(k_i)$ denote a collection of pairwise disjoint regular neighborhoods one containing each arc of k_i such that $N(k_i) \subseteq N(B_i)$. Let $V_i = \mathrm{cl}(W_i - (N(C_i) \cup N(J_i) \cup N(B_i) \cup N(e_i)))$, let $\Delta = \mathrm{cl}(N(B_i) - N(k_i))$, and let $V_i' = V_i \cup \Delta$. Since $N(B_i)$ is a solid annulus, it has a product structure $D^2 \times I$. Without loss of generality, we assume that each of the components of $N(k_i)$ respects the product structure of $N(B_i)$. Thus $\Delta = F \times I$ where F is a disk with holes.

Definition 2. *Let X be a 3-manifold. A sphere in X is said to be* **essential** *if it does not bound a ball in X. A properly embedded disk D in X is said to be* **essential** *if ∂D does not bound a disk in ∂X. A properly embedded annulus is said to be* **essential** *if it is incompressible and not boundary parallel. A torus in X is said to be* **essential** *if it is incompressible and not boundary parallel.*

Lemma 1. *For each i, V_i' contains no essential torus, sphere, or disk whose boundary is in $D_i \cup D_{i+1}$. Also, any incompressible annulus in V_i' whose boundary is in $D_i \cup D_{i+1}$ is either boundary parallel or can be expressed as $\sigma \times I$ (possibly after a change in parameterization of Δ), where σ is a non-trivial simple closed curve in $D_i \cap \Delta$.*

Proof. Since k_i contains at least three disjoint arcs, F is a disk with at least three holes. Let β denote the double of Δ along $\Delta \cap (D_i \cup D_{i+1})$. Then $\beta = F \times S^1$. Now it follows from Waldhausen [7] that β contains no essential sphere or properly embedded disk, and any incompressible torus in β can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of β) where σ is a non-trivial simple closed curve in $D_i \cap \Delta$.

Let ν denote the double of V_i along $V_i \cap (D_i \cup D_{i+1})$. Observe that $\nu \cup \beta$ is the double of V_i' along $V_i' \cap (D_i \cup D_{i+1})$. Now the interior of ν is homeomorphic to $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$. Since we saw above that $M_i - (\overline{Q}_1 \cup \overline{Q}_2 \cup j \cup b \cup E)$ is hyperbolic, it follows from Thurston [5, 6] that ν contains no essential sphere or torus, or properly embedded disk or annulus.

We see as follows that $\nu \cup \beta$ contains no essential sphere and any essential torus in $\nu \cup \beta$ can be expressed (after a possible change in parameterization of β) as $\sigma \times S^1$, where σ is a non-trivial simple closed curve in $D_i \cap \Delta$. Let

τ be an essential sphere or torus in $\nu \cup \beta$, and let γ denote the torus $\nu \cap \beta$. By doing an isotopy as necessary, we can assume that τ intersects γ in a minimal number of disjoint simple closed curves. Suppose there is a curve of intersection which bounds a disk in the essential surface τ. Let c be an innermost curve of intersection on τ which bounds a disk δ in τ. Then δ is a properly embedded disk in either γ or β. Since neither ν nor β contains a properly embedded essential disk or an essential sphere, there is an isotopy of τ which removes c from the collection of curves of intersection. Thus by the minimality of the number of curves in $\tau \cap \gamma$, we can assume that none of the curves in $\tau \cap \gamma$ bounds a disk in τ.

Suppose that τ is an essential sphere in $\nu \cup \beta$. Since none of the curves in $\tau \cap \gamma$ bounds a disk in τ, τ must be contained entirely in either ν or β. However, we saw above that neither ν nor β contains any essential sphere. Thus τ cannot be an essential sphere, and hence must be an essential torus. Since $\tau \cap \gamma$ is minimal, if $\tau \cap \nu$ is non-empty, then the components of τ in ν are all incompressible annuli. However, we saw above that ν contains no essential annuli. Thus $\tau \cap \nu$ is empty. Since ν contains no essential torus, the essential tori τ must be contained in β. Hence τ can be expressed (after a possible change in parameterization of β) as $\sigma \times S^1$, where σ is a non-trivial simple closed curve in $D_i \cap \Delta$.

Now we consider essential surfaces in V_i'. Suppose that V_i' contains an essential sphere S. Since $\nu \cap \beta$ contains no essential sphere, S bounds a ball B in $\nu \cap \beta$. Now the ball B cannot contain any of the boundary components of $\nu \cap \beta$. Thus B cannot contain either D_i or D_{i+1}. Since S is disjoint from $D_i \cup D_{i+1}$, it follows that B must be disjoint from $D_i \cup D_{i+1}$. Thus B is contained in V_i'. Hence V_i' cannot contain an essential sphere.

We see as follows that V_i' cannot contain an essential disk whose boundary is in $D_i \cup D_{i+1}$. Let ϵ be a disk in V_i' whose boundary is in $D_i \cup D_{i+1}$. Let ϵ' denote the double of ϵ in $\nu \cup \beta$. Then ϵ' is a sphere which meets $D_i \cup D_{i+1}$ in the simple closed curve $\partial\epsilon$. Since $\nu \cup \beta$ contains no essential sphere, ϵ' bounds a ball B in $\nu \cup \beta$. It follows that B cannot contain any of the boundary components of $\nu \cup \beta$. Thus B cannot contain any of the boundary components of $D_i \cup D_{i+1}$. Therefore, $D_i \cup D_{i+1}$ intersects the ball B in a disk bounded by $\partial\epsilon$. Hence the simple closed curve $\partial\epsilon$ bounds a disk in $(D_i \cup D_{i+1}) \cap V_i'$, and therefore the disk ϵ was not essential in V_i'. Thus, V_i' contains no essential disk whose boundary is in $D_i \cup D_{i+1}$.

Now suppose that V_i' contains an essential torus T. Suppose that T is not essential in $\nu \cup \beta$. Then either T is boundary parallel or T is compressible in $\nu \cup \beta$. However, T cannot be boundary parallel in $\nu \cup \beta$ since $T \subseteq V_i'$. Thus T must be compressible in $\nu \cup \beta$. Let δ be a compression disk for T in $\nu \cup \beta$. Since V_i' contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$, we can use an innermost disk argument to push δ off of $D_i \cup D_{i+1}$. Hence T is compressible in V_i', contrary to our initial assumption. Thus T must be essential in $\nu \cup \beta$. It follows that T has the form $\sigma \times S^1$, where $\sigma \subseteq D_i \cap \Delta$. However, since $\nu \cup \beta$ is the double of V_i', the intersection of

$\sigma \times S^1$ with V_i' is an annulus $\sigma \times I$. In particular, V_i' cannot contain $\sigma \times S^1$. Therefore, V_i' cannot contain an essential torus.

Suppose that V_i' contains an incompressible annulus α whose boundary is in $D_i \cup D_{i+1}$. Let τ denote the double of α in $\nu \cup \beta$. Then τ is a torus. If τ is essential in $\nu \cup \beta$, then we saw above that τ can be expressed as $\sigma \times S^1$ (after a possible change in parameterization of β) where σ is a non-trivial simple closed curve in $D_i \cap \Delta$. In this case, α can be expressed as $\sigma \times I$.

On the other hand, if τ is inessential in $\nu \cup \beta$, then either τ is parallel to a component of $\partial(\nu \cup \beta)$, or τ is compressible in $\nu \cup \beta$. If τ is parallel to a boundary component of $\nu \cup \beta$, then α is parallel to the annulus boundary component of W_i, $N(J_i)$, $N(e_i)$, $N(B_i)$, or one of the boundary components of $N(k_i)$.

Thus we suppose that the torus τ is compressible in $\nu \cup \beta$. In this case, it follows from an innermost loop outermost arc argument that either the annulus α is compressible in V_i' or α is ∂-compressible in V_i'. Since we assumed α was incompressible in V_i', α must be ∂-compressible in V_i'. Now according to a lemma of Waldhausen [7], if a 3-manifold contains no essential sphere or properly embedded essential disk, then any annulus which is incompressible but boundary compressible must be boundary parallel. We saw above that V_i' contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$. Since the boundary of the incompressible annulus α is contained in $D_i \cup D_{i+1}$, it follows from Waldhausen's Lemma that α is boundary parallel in V_i'. □

It follows from Lemma 1 that for any i, any incompressible annulus in V_i' whose boundary is in $D_i \cup D_{i+1}$ is either parallel to an annulus in D_i or D_{i+1} or co-bounds a solid annulus in the solid annulus W_i with ends in $D_i \cup D_{i+1}$. Recall that κ is a simple closed curve in Γ_1 such that $\kappa \cap W_i = k_i \cup e_i$. Also $J = J_1 \cup \cdots \cup J_n$. Let $N(\kappa)$ and $N(J)$ be regular neighborhoods of the simple closed curves κ and J respectively, such that for each i, $N(\kappa) \cap W_i = N(k_i) \cup N(e_i)$, and $N(J) \cap W_i = N(J_i)$. Recall that $V = W_1 \cup \cdots \cup W_n$. Thus $\mathrm{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J) \cup N(\kappa)) = V_1' \cup \cdots \cup V_n'$.

Proposition 1. $H = \mathrm{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J) \cup N(\kappa))$ *contains no essential sphere or torus.*

Proof. Suppose that S is an essential sphere in H. Without loss of generality, S intersects the collection of disks $D_1 \cup \cdots \cup D_n$ transversely in a minimal number of simple closed curves. By Lemma 1, for each i, V_i' contains no essential sphere or essential disk whose boundary is in $D_i \cup D_{i+1}$. Thus the sphere S cannot be entirely contained in one V_i'. Let c be an innermost curve of intersection on S. Then c bounds a disk δ in some V_i'. However, since the number of curves of intersection is minimal, δ must be essential, contrary to Lemma 1. Hence H contains no essential sphere.

Suppose T is an incompressible torus in H. We show as follows that T is parallel to some boundary component of H. Without loss of generality, the torus T intersects the collection of disks $D_1 \cup \cdots \cup D_n$ transversely in a minimal number of simple closed curves. By Lemma 1, for each i, V_i' contains no essential torus, essential sphere, or essential disk whose boundary is in $D_i \cup D_{i+1}$. Thus the torus T cannot be entirely contained in one V_i'. Also, by the minimality of the number of curves of intersection, we can assume that if $V_i' \cap T$ is nonempty, then it consists of a collection of incompressible annuli in V_i' whose boundary components are in $D_i \cup D_{i+1}$. Furthermore, by Lemma 1, each such annulus is either boundary parallel or is contained in $N(B_i)$ and can be expressed (after a possible change in parameterization of $N(B_i)$) as $\sigma_i \times I$ for some non-trivial simple closed curve σ_i in $D_i \cap \Delta$. If some annulus component of $V_i' \cap T$ is parallel to an annulus in $D_i \cup D_{i+1}$, then we could remove that component by an isotopy of T. Thus we can assume that each annulus in $V_i' \cap T$ is parallel to the annulus boundary component of one of the solid annuli W_i, $N(J_i)$, or $N(e_i)$, or can be expressed as $\sigma_i \times I$. In any of these cases the annulus co-bounds a solid annulus in W_i with ends in $D_i \cup D_{i+1}$.

Consider some i, such that $V_i' \cap T$ is non-empty. Hence it contains an incompressible annulus A_i which has one of the above forms. By the connectivity of the torus T, either there is an incompressible annulus $A_{i+1} \subseteq V_{i+1}' \cap T$ such that A_i and A_{i+1} share a boundary component, or there is an incompressible annulus $A_{i-1} \subseteq V_{i-1}' \cap T$, such that A_i and A_{i-1} share a boundary component, or both. We will assume, without loss of generality, that there is an incompressible annulus $A_{i+1} \subseteq V_{i+1}' \cap T$ such that A_i and A_{i+1} share a boundary component. Now it follows that A_i co-bounds a solid annulus F_i in W_i with ends in $D_i \cup D_{i+1}$, and A_{i+1} co-bounds a solid annulus F_{i+1} in W_{i+1} together with two disks in $D_{i+1} \cup D_{i+2}$. Hence the solid annuli F_i and F_{i+1} meet in one or two disks in D_{i+1}.

We consider several cases where A_i is parallel to some boundary component of V_i'. Suppose that A_i is parallel to the annulus boundary component of the solid annulus $N(J_i)$. Then the solid annulus F_i contains $N(J_i)$ and is disjoint from the arcs k_i and e_i. Now the arcs J_i and J_{i+1} share an endpoint contained in $F_i \cap F_{i+1}$, and there is no endpoint of any arc of k_i or e_i in $F_i \cap F_{i+1}$. It follows that the solid annulus F_{i+1} contains the arc J_{i+1} and contains no arcs of k_{i+1}. Hence by Lemma 1, the incompressible annulus A_{i+1} must be parallel to $\partial N(J_{i+1})$. Continuing from one V_i' to the next, we see that in this case, T is parallel to $\partial N(J)$.

Suppose that A_i is parallel to the annulus boundary component of the solid annulus $\partial N(e_i)$ or one of the solid annuli in $\partial N(k_i)$. Using an argument similar to the above paragraph, we see that A_{i+1} is parallel to the annulus boundary component of the solid annulus $\partial N(e_{i+1})$ or one of the solid annuli in $\partial N(k_{i+1})$. Continuing as above, we see that in this case T is parallel to $\partial N(\kappa)$.

Suppose that the annulus A_i is parallel to the annulus boundary component of the solid annulus W_i. Then the solid annulus F_i contains all of the arcs of J_i, k_i, and e_i. It follows as above that the solid annulus F_{i+1} contains the arc J_{i+1} and some arcs of $k_{i+1} \cup e_{i+1}$. Thus by Lemma 1, A_{i+1} must be parallel to the annulus boundary component of the solid annulus W_{i+1}. Continuing in this way, we see that in this case T is parallel to ∂V.

Thus we now assume that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of V_i'. Hence if any $V_i' \cap T$ is non-empty, then by Lemma 1, $V_i' \cap T$ consists of disjoint incompressible annuli in $N(B_i)$ which can each be expressed (after a possible re-parametrization of $N(B_i)$) as $\sigma_i \times I$ for some non-trivial simple closed curve $\sigma_i \subseteq D_i \cap \Delta$. Choose i such that $V_i' \cap T$ is non-empty. Since $N(B_i)$ is a solid annulus, there is an innermost incompressible annulus A_i of $N(B_i) \cap T$. Now A_i bounds a solid annulus F_i in $N(B_i)$, and F_i contains more than one arc of k_i. Since A_i is innermost in $N(B_i)$, $\mathrm{int}(F_i)$ is disjoint from T. Now there is an incompressible annulus A_{i+1} in $V_{i+1}' \cap T$, such that A_i and A_{i+1} meet in a circle in D_{i+1}. Furthermore, this circle bounds a disk in D_{i+1} which is disjoint from T, and by our assumption is contained in $N(B_i)$. Thus by Lemma 1, the incompressible annulus A_{i+1} has the form $\sigma_{i+1} \times I$ for some non-trivial simple closed curve $\sigma_{i+1} \subseteq D_{i+1} \cap \Delta$. Thus A_{i+1} bounds a solid annulus F_{i+1} in $N(B_{i+1})$, and $\mathrm{int}(F_{i+1})$ is also disjoint from T. We continue in this way considering consecutive annuli to conclude that for every j, every component A_j of $T \cap V_j'$ is an incompressible annulus which bounds a solid annulus F_j whose interior is disjoint from T.

Recall that $V = W_1 \cup \cdots \cup W_n$ is a solid torus. Let Q denote the component of $V - T$ which is disjoint from ∂V. Then Q is the union of the solid annuli F_j. Since some F_i contains some arcs of k_i, the simple closed curve κ must be contained in Q.

Recall that the simple closed curve κ contains at least three vertices of the embedded graph Γ_1. Also each vertex of κ is contained in some arc e_j. Since each such $e_j \subseteq \kappa \subseteq Q$, some component F_j of $Q \cap W_j$ contains the arc e_j. By our assumption, for any $V_i' \cap T$ which is non-empty, $V_i' \cap T$ consists of disjoint incompressible annuli in $N(B_i)$. In particular, $V_j \cap T \subseteq N(B_i)$. Now the annulus boundary of F_j is contained in $N(B_j)$, and hence $F_j \subseteq N(B_j)$. But this is impossible since $e_j \subseteq F_j$ and e_j is disjoint from $N(B_j)$. Hence our assumption that no component of any $V_i' \cap T$ is parallel to an annulus boundary component of V_i' is wrong. Thus, as we saw in the previous cases, T must be parallel to a boundary component of H. Therefore H contains no essential annulus. □

Recall that the value of r, the simple closed curves, and the manifold H, all depend on the particular choice of simple closed curve κ. In the following theorem, we do not fix a particular κ, so none of the above are fixed.

Theorem 1. *Every graph can be embedded in S^3 in such a way that every non-trivial knot in the embedded graph is hyperbolic.*

Proof. Let G be a graph, and let $n \geq 3$ be an odd number such that G is a minor of the complete graph on n vertices K_n. Let Γ_1 be the embedding of K_n given in our preliminary construction. Then, Γ_1 contains at most finitely many simple closed curves, $\kappa_1, \ldots, \kappa_m$. For each κ_j, we use Thurston's Hyperbolic Dehn Surgery Theorem [1, 5] to choose an r_j in the same manner that we chose r after we fixed a particular simple closed curve κ. Now let $R = \max\{r_1, \ldots, r_m\}$, and let R be the value of r in Figure 1. This determines the simple closed curves $C_1, \ldots, C_n$.

Let $P = \mathrm{cl}(V - (N(C_1) \cup \cdots \cup N(C_n) \cup N(J)))$ where V and J are given in our preliminary construction. Then the embedded graph $\Gamma_1 \subseteq P$. For each $j = 1, \ldots m$, let $H_j = \mathrm{cl}(P - N(\kappa_j))$. It follows from Proposition 1 that each H_j contains no essential sphere or torus. Since each H_j has more than three boundary components, no H_j can be Seifert fibered. Hence by Thurston's Hyperbolization Theorem [6], every H_j is a hyperbolic manifold.

We will glue solid tori $Y_1, \ldots, Y_{n+2}$ to P along its $n+2$ boundary components ∂V, $\partial N(C_1), \ldots, \partial N(C_n)$, and $\partial N(J)$ to obtain a closed manifold $\overline{P}$ as follows. For each j, any gluing of solid tori along the boundary components of P defines a Dehn filling of $H_j = \mathrm{cl}(P - N(\kappa_j))$ along all of its boundary components except $\partial N(\kappa_j)$. Since each H_j is hyperbolic, by Thurston's Hyperbolic Dehn Surgery Theorem [1, 5], all but finitely many such Dehn fillings of H_j result in a hyperbolic 3-manifold. Furthermore, since P is obtained by removing solid tori from S^3, for any integer q, if we attach the solid tori $Y_1, \ldots, Y_{n+2}$ to P with slope $\frac{1}{q}$, then $\overline{P} = S^3$. In this case each $H_j \cup Y_1 \cup \cdots \cup Y_{n+2}$ is the complement of a knot in S^3. There are only finitely many H_j's, and for each j, only finitely many slopes $\frac{1}{q}$ are excluded by Thurston's Hyperbolic Dehn Surgery Theorem. Thus there is some integer q such that if we glue the solid tori $Y_1, \ldots, Y_{n+2}$ to any of the H_j along $\partial N(C_1), \ldots, \partial N(C_n)$, $\partial N(J)$, ∂V with slope $\frac{1}{q}$, then we obtain the complement of a hyperbolic knot in S^3.

Let Γ_2 denote the re-embedding of Γ_1, obtained as a result of gluing the solid tori $Y_1, \ldots, Y_{n+2}$ to the boundary components of P with slope $\frac{1}{q}$. Now Γ_2 is an embedding of K_n in S^3 such that every non-trivial knot in Γ_2 is hyperbolic. Now there is a minor G' of the embedded graph Γ_2 which is an embedding of our original graph G, such that every non-trivial knot in G' is hyperbolic. □

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