EXTENSION OF INCOMPRESSIBLE SURFACES ON THE BOUNDARY OF 3-MANIFOLDS

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Abstract. An incompressible surface F on the boundary of a compact orientable 3-manifold M is arc-extendible if there is an arc γ on $\partial M - \text{Int}F$ such that $F \cup N(\gamma)$ is incompressible, where $N(\gamma)$ is a regular neighborhood of γ in ∂M. Suppose for simplicity that M is irreducible, and F has no disk components. If M is a product $F \times I$, or if $\partial M - F$ is a set of annuli, then clearly F is not arc-extendible. The main theorem of this paper shows that these are the only obstructions for F to be arc-extendible.

Suppose F is a compact incompressible surface on the boundary of a compact, orientable, irreducible 3-manifold M. Let F' be a component of $\partial M - \text{Int}F$. We say that F is *arc-extendible* (in F') if there is a properly embedded arc γ in F' such that $F \cup N(\gamma)$ is incompressible. In this case γ is called an *extension arc* of F. We study the problem of which incompressible surfaces on the boundary M are arc-extendible. This is useful in, for example, finding a sequence of mutually nonparallel incompressible surfaces in a 3-manifold.

Denote by I the unit interval $[0,1]$. We say that M is a product $F \times I$ if there is a homeomorphism $\varphi : M \cong F \times I$ with $\varphi(F) = F \times 1$. Note that in this case $F' = \partial M - \text{Int}F$, and F is not arc-extendible. A surface F is *diskless* if it has no disk component. An incompressible surface with a disk component is always arc-extendible, unless the disk lies on a sphere component of ∂M. Thus to avoid trivial cases, we will only consider arc-extension of diskless surfaces.

Theorem 1. *Let F be a diskless, compact, incompressible surface on the boundary of a compact, orientable, irreducible 3-manifold M, and let F' be a non-annular component of $\partial M - \text{Int}F$. Then either F is arc-extendible in F', or M is a product $F \times I$.*

The proof of the theorem involve some deep results about incompressible surfaces related to Dehn surgery and 2-handle additions. It breaks down into three cases. The

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case that F' is a thrice punctured sphere is treated in Theorem 4, which shows that if the surface obtained by gluing F and F' along one of the boundary curve of F' is compressible for all the three boundary curves of F', then M must be a product. The second case is that F' is parallel into F (see below for definition). A similar result as above holds in this case. Theorem 9 shows that in the remaining case there is an arc γ intersecting some circle C in F' at one point, so that all but at most three Dehn twists of γ along C are extension arcs of F. Moreover, in this case the extension arc γ of F can be chosen to have endpoints on any prescribed components of $\partial F'$. See Theorem 10 below.

Note that the irreducibility of M is irrelevant to the compressibility of surfaces on ∂M. However, this does make the conclusion of the theorem simpler. If we drop this assumption from the theorem, the conclusion should be changed to "Either F is arc-extendible in F', or there is a component F_0 of F, and a homeomorphism $\varphi : M \cong F_0 \times I \# M'$ for some M', such that $\varphi(F_0) = F_0 \times 1$, and $\varphi(F') = F_0 \times 0 \cup \partial F_0 \times I$."

Given a simple closed curve α on a surface S on the boundary of M, we use $M[\alpha]$ to denote the manifold obtained by adding a 2-handle to M along the curve α. More explicitly, $M[\alpha]$ is the union of M and a $D^2 \times I$, with the annulus $(\partial D^2) \times I$ glued to a regular neighborhood $N(\alpha)$ of α on ∂M. Use $S[\alpha]$ to denote the surface in $M[\alpha]$ corresponding to S, i.e. $S[\alpha] = (S - N(\alpha)) \cup (D^2 \times \partial I)$. The following two lemmas are very useful in dealing with incompressible surfaces. Various versions of Lemma 2 have been proved by Przytycki [Pr], Johannson [Jo], Jaco [Ja], and Scharlemann [Sch]. The lemma as stated is due to Casson and Gordon [CG].

Lemma 2. (The Handle Addition Lemma [CG].) *Let α be a simple closed curve on a surface S on the boundary of an orientable irreducible 3-manifold M, such that S is compressible and $S - \alpha$ is incompressible. Then $S[\alpha]$ is incompressible in $M[\alpha]$, and $M[\alpha]$ is irreducible.*

Lemma 3. (The Generalized Handle Addition Lemma.) *Let S be a surface on the boundary of an orientable 3-manifold M, let γ be a 1-manifold on S, and let α be a circle on S disjoint from γ. Suppose $S - \gamma$ is compressible and $S - (\gamma \cup \alpha)$ is incompressible. If D is a compressing disk of $S[\alpha]$ in $M[\alpha]$, then there is a compressing disk D' of $S - \alpha$ in M such that $\partial D' \cap \gamma \subset \partial D \cap \gamma$.*

Proof. This is essentially [Wu2, Theorem 1]. The theorem there stated that $\partial D' \cap \gamma$ has no more points than $\partial D \cap \gamma$, but the proof there gives the stronger conclusion that $\partial D' \cap \gamma \subset \partial D \cap \gamma$. □

We first study the case that the surface F' in Theorem 1 is a thrice punctured sphere. Let $\alpha_1, \alpha_2, \alpha_3$ be the boundary curves of F'. Since F' is a component of

$\partial M - \mathrm{Int}F$, we have $\alpha_i \subset \partial F$ for $i = 1, 2, 3$. Note that if $\mathrm{Int}F \cup \mathrm{Int}F' \cup \alpha_i$ is incompressible for some i, then for any essential arc γ on F' with $\partial\gamma \subset \alpha_i$, the surface $F \cup N(\gamma)$ is incompressible. Hence the following theorem proves Theorem 1 in the case that F' is a twice punctured disk. However, it should be noticed that a similar statement is false if we drop the assumption that F' is a sphere with three holes.

Theorem 4. *Let F be a diskless compact incompressible surface on the boundary of a compact, orientable, irreducible 3-manifold M, and let F' be a component of $\partial M - \mathrm{Int}F$ which is a punctured sphere with $\partial F' = \alpha_1 \cup \alpha_2 \cup \alpha_3$. If $\mathrm{Int}F \cup \mathrm{Int}F' \cup \alpha_i$ is compressible for $i = 1, 2, 3$, then M is a product $F \times I$.*

Proof. We fix some notation. Write $\widehat{F} = F \cup F'$. Denote by $\widehat{F}_i$ the surface obtained by gluing $\mathrm{Int}F$ and $\mathrm{Int}F'$ along α_i, i.e. $\widehat{F}_i = \mathrm{Int}F \cup \mathrm{Int}F' \cup \alpha_i$. Similarly, write $\widehat{F}_{ij} = \mathrm{Int}F \cup \mathrm{Int}F' \cup \alpha_i \cup \alpha_j$.

First notice that F' is incompressible. This is because each simple closed curve on F' is isotopic to one of the $\alpha_i \subset F$, and because F is incompressible and diskless. Since $\mathrm{Int}F \cap \mathrm{Int}F' = \emptyset$, the surface $\mathrm{Int}F \cup \mathrm{Int}F'$ is incompressible.

Let M' be a maximal compression body of ∂M in M. Then a surface on the boundary of M is compressible in M if and only if it is compressible in M'. Notice that if $M \neq M'$, then M' is never a product $F \times I$, so if the theorem is true for M', it is true for M. Hence after replacing M by M' if necessary, we may assume without loss of generality that M is a compression body.

We claim that the curves $\alpha_1, \alpha_2, \alpha_3$ are mutually nonparallel on $\widehat{F}$, that is, no component of F is an annulus with both boundary components on F'. If two curves α_1, α_2, say, are parallel on $\widehat{F}$, then the surface $\mathrm{Int}F \cup \mathrm{Int}F' = \widehat{F} - \alpha_1 \cup \alpha_2 \cup \alpha_3$ is incompressible if and only if $\widehat{F}_1 = \widehat{F} - \alpha_2 \cup \alpha_3$ is incompressible. However, by assumption $\widehat{F}_1$ is compressible, and we have shown that $\mathrm{Int}F \cup \mathrm{Int}F'$ is incompressible. Hence the claim follows.

Since $\widehat{F}_i$ is compressible, and $\widehat{F}_i - \alpha_i = \mathrm{Int}F \cup \mathrm{Int}F'$ is incompressible, we can apply the Handle Addition Lemma (Lemma 2) to $\widehat{F}_i$ and α_i to conclude that after adding a 2-handle along α_i, the surface $\widehat{F}_i[\alpha_i]$ is incompressible in $M[\alpha_i]$, and $M[\alpha_i]$ is irreducible.

Consider the surface $\widehat{F}[\alpha_1]$. Notice that after adding the 2-handle, the surface F' becomes an annulus on $\widehat{F}[\alpha_1]$ with boundary $\alpha_2 \cup \alpha_3$, so the two curves α_2, α_3 are parallel on $\widehat{F}[\alpha_1]$. Thus, $\widehat{F}_1[\alpha_1] = \widehat{F}[\alpha_1] - \alpha_2 \cup \alpha_3$ being incompressible in $M[\alpha_1]$ implies that $\widehat{F}[\alpha_1] - \alpha_2$ is incompressible in $M[\alpha_1]$. With the above notation, this says that $\widehat{F}_{13}[\alpha_1]$ is incompressible in $M[\alpha_1]$.

By assumption $\widehat{F}_3$ is compressible in M. Let D be a compressing disk of $\widehat{F}_3$ in M. Then ∂D is disjoint from $\alpha_1 \cup \alpha_2$, because $\partial D \subset \widehat{F}_3$. Also, ∂D is not isotopic to α_1 in

$\widehat{F}_{13}$, otherwise α_1 would bound a disk in M, contradicting the assumption that F is diskless and incompressible. We have shown that $\widehat{F}_{13}[\alpha_1]$ is incompressible in $M[\alpha_1]$, so D is not a compressing disk of $\widehat{F}_{13}[\alpha_1]$ in $M[\alpha_1]$, and hence ∂D must bound a disk in $\widehat{F}_{13}[\alpha_1]$. This is true if and only if ∂D is coplanar to α_1 on $\widehat{F}_{13}$, that is, either ∂D is parallel to α_1, or it bounds a once punctured torus T in $\widehat{F}_{13}$ which contains α_1 as a nonseparating curve. The first possibility has been ruled out, so the second must be true. Let $\widehat{T}$ be the torus $T \cup D$. Since we have assumed above that M is a compression body, either (i) $\widehat{T}$ is parallel to a boundary component of M, or (ii) $\widehat{T}$ bounds a solid torus.

If $\widehat{T}$ is parallel to a boundary component T_0 of M, then after adding the 2-handle, the surface $\widehat{T}[\alpha_1]$ becomes a sphere which separates T_0 from $\widehat{F}[\alpha_1]$, hence is a reducing sphere of $M[\alpha_1]$, which contradicts the irreducibility of $M[\alpha_1]$. Similarly, if $\widehat{T}$ bounds a solid torus V but α_1 is not a longitude of V, then after adding the 2-handle the manifold would have a lens space or $S^2 \times S^1$ summand, which again contradicts the irreducibility of $M[\alpha_1]$. (Note that $M[\alpha_1]$ cannot be a lens space because it has some boundary components.)

We have now shown that there is a compressing disk D of $\widehat{F}_3$ in M which cuts the manifold into two pieces, one of which is a solid torus V which contains α_1 as a longitude, but is disjoint from α_2. Let D_1 be a meridian disk of V. Then $\partial D_1 \cap \alpha_1$ is a single point, and ∂D_1 is disjoint from α_2 because ∂V is disjoint from α_2. Notice that ∂D_1 is not coplanar to α_2, for if ∂D_1 were parallel to α_2 then α_2 would also intersect α_1, and if ∂D_1 would bound a once punctured torus containing α_2 then ∂D_1 would be a separating curve on ∂M, so it would intersect α_1 in an even number of points, either case leading to a contradiction. Thus, after adding a 2-handle to M along α_2, the disk D_1 remains a compressing disk of $\widehat{F}[\alpha_2]$. Since the two curves α_1 and α_3 are parallel in $\widehat{F}[\alpha_2]$, and since D_1 intersects α_1 in a single point, we can isotope D_1 to another disk D_2 in $M[\alpha_2]$ so that it intersects each of α_1 and α_3 in a single point. We are looking for such a disk in M; however D_2 is not necessary the one because it may intersect the attached 2-handle.

Recall that the surface $\widehat{F}_2$ is compressible, but the surface $\widehat{F}_2 - \alpha_2 = \mathrm{Int}F \cup \mathrm{Int}F'$ is incompressible. Hence we can apply the Generalized Handle Addition Lemma (Lemma 3, with $S = \widehat{F}$, $\gamma = \alpha_1 \cup \alpha_3$, and $\alpha = \alpha_2$) to conclude that there is also a compressing disk D_3 of $\widehat{F}$ in M, such that ∂D_3 is disjoint from α_2, and $\partial D_3 \cap (\alpha_1 \cup \alpha_3)$ is a subset of $\partial D_2 \cap (\alpha_1 \cup \alpha_3)$.

The set $\partial D_3 \cap (\alpha_1 \cup \alpha_3)$ is nonempty, otherwise, since ∂D_3 is also disjoint from α_2, D_3 would be a compressing disk of $\mathrm{Int}F \cup \mathrm{Int}F'$, contradicting the incompressibility of $\mathrm{Int}F \cup \mathrm{Int}F'$. Since $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is separating on $\widehat{F}$, the curve ∂D_3 can not intersect $\alpha_1 \cup \alpha_2 \cup \alpha_3$ at a single point. It follows that $\partial D_3 \cap (\alpha_1 \cup \alpha_3) = \partial D_2 \cap (\alpha_1 \cup \alpha_3)$, that is, ∂D_3 intersects each of α_1, α_3 in a single point. Such a disk is called a *bigon.*

Denote by D_{13} the bigon D_3 above. Interchanging the rules of α_1 and α_2 in the above argument, we get another compressing disk D_{23} of $\widehat{F}$ in M, which is disjoint from α_1, and intersects each of α_2, α_3 in a single point. By a simple innermost circle outermost arc argument, we can isotope D_{13} so that it is disjoint from D_{23}, and still has the same number of intersection points with each α_i. Cutting M along $D_{13} \cup D_{23}$, we get a submanifold M' of M, in which the surface F' becomes a disk $\widetilde{F}' \subset F'$, and the surface F becomes a surface $\widetilde{F} \subset F$. It is clear that one boundary component C of $\widetilde{F}$ bounds a disk on $\partial M'$, namely the union of $\widetilde{F}'$ and the two copies of $D_{13} \cup D_{23}$. Since F is incompressible, this curve C bounds a disk in F, so $\widetilde{F}$ must be a disk. These disks together form a sphere boundary component of M'. Since M is irreducible, M' must be a 3-ball, so it is a product $\widetilde{F} \times I$. Gluing back along D_{13} and D_{23}, we see that M is a product $F \times I$. This completes the proof of Theorem 4. $\square$

Below, F, F' and M will be as in Theorem 1. Using Theorem 4 we may assume that F' is not a thrice punctured sphere. A curve C' on F' is *∂-nonseparating* if (i) C' is not parallel to a boundary curve on F', and (ii) there is a proper arc γ in F' intersecting C' in a single point. A sub-surface G' of F' is *parallel into* F if there is a product $G' \times I \subset M$ such that $G' = G' \times 0$, and $G' \times 1 \subset F$. Similarly, a curve C' on F' is *parallel into* F if there is an embedded annulus $A \subset M$ with $\partial A = C' \cup C$, where $C \subset F$.

Lemma 5. *If F' is compressible, then there is a ∂-nonseparating curve C' on F' which is not parallel into F.*

Proof. Let D be a compressing disk of F'. If ∂D is non-separating on F', let C' be a curve in F' that intersects ∂D in one point. Then C' is nonseparating, hence ∂-nonseparating on F'. We want to show that C' is not parallel into F. Otherwise, let A be an annulus with $\partial A = C' \cup C$, where $C \subset F$. Then $A \cap D$ is a proper 1-manifold on D. But $\partial(A \cap D) = (\partial A) \cap \partial D$ is a single point, which is absurd. Hence C' is the curve required.

Now assume that ∂D is separating on F', cutting F' into F'_1 and F'_2. Choose a simple loop C_i on F'_i as follows. If F'_i is nonplanar, then there are a pair of non-separating curves intersecting each other in one point, at least one of which is not null-homologous in M. Choose this one as C_i. If F'_i is planar, choose C_i to be isotopic to a boundary curve of F'. Note that since F is incompressible and diskless, C_i is not null-homotopic in M. Also notice that in both cases there is a properly embedded arc γ on one of the F'_i which intersects $C_1 \cup C_2$ in one point.

Now choose a band $B = I \times I$ on F' such that $B \cap \partial D = I \times \frac{1}{2}$, $B \cap C_1 = I \times 0$, $B \cap C_2 = I \times 1$, and B is disjoint from the arc γ above. Such band exists because

γ is a nonseparating arc on F'_i. Let C' be the band sum of C_1 and C_2, that is, $C' = (C_1 \cup C_2 - I \times \{0,1\}) \cup (\{0,1\} \times I)$. Then C' intersects γ in one point. Since C' intersects ∂D essentially in two points, it is not parallel to any boundary component on F'. Therefore C' is ∂-nonseparating.

We want to show that C' is not parallel into F. Using the property that C_i are not null-homotopic in M, one can show by an innermost circle argument that C' is not null-homotopic in M. Now suppose that there is an annulus A in M with $\partial A = C' \cup C$, where $C \subset F$. Since C' is not null-homotopic in M, A is incompressible in M. By surgery along an innermost circle of $D \cap A$ one can eliminate all circle intersections of $A \cap D$. Since $\partial(A \cap D)$ consists of two points, $A \cap D$ is a single arc, which has endpoints on the same component of ∂A, hence it cuts off a disk D' from A. Assume without loss of generality that $D' \cap F'$ is on F'_1. Let D'' be the disk on D bounded by $(A \cap D) \cup (B \cap D)$, and let $B_1 = B \cap F'_1$. Then $D' \cup D'' \cup B_1$ is a disk with boundary C_1, which contradicts the fact that C_1 is not null-homotopic in M. Therefore, C' is not parallel into F. $\square$

Lemma 6. *Suppose F' is incompressible, and is not a thrice punctured sphere. Then either (i) there is a ∂-nonseparating curve C' on F' which is not parallel into F, or (ii) F' is parallel into F.*

Proof. Since F' is not a thrice punctured sphere, one can easily find a ∂-nonseparating curve α_0 on F'. Assume that (i) is not true, so all ∂-nonseparating curves are parallel into F. We want to show that F' is parallel into F.

Since α_0 is parallel into F, the annulus $N(\alpha_0)$ is also parallel into F. It is an incompressible annulus because α_0 is essential on F' and F' is incompressible. Among all connected incompressible surfaces in $\text{Int}F'$ which contain α_0 and are parallel into F, choose G' such that the complexity $(\chi(G'), |\partial G'|)$ is minimal in the lexical-graphic order, where $\chi(G')$ is the Euler characteristic of G', and $|\partial G'|$ is the number of boundary components of G'. All incompressible sub-surfaces of F' have Euler characteristics bounded below by $\chi(F')$, hence such G' does exist.

If all boundary components of G' are parallel to some boundary components on F', then either G' is contained in a collar of $\partial F'$, or $F' - \text{Int}G' = \partial F' \times I$. The first case does not happen because G' contains the ∂-nonseparating curve α_0, which by definition is not parallel to any boundary curve on F'. In the second case F' is isotopic to G', so it is parallel into F, and we are done. Hence we may assume that some boundary curve β of G' is not parallel to any boundary curve on F'.

We want to find a ∂-nonseparating curve α' which intersects β essentially in one or two points. If β is nonseparating on F', choose α' to be any curve on F' that intersects β in a single point. Then α' is nonseparating, hence ∂-nonseparating on F'.

If β separates F' into F'_1 and F'_2, choose an essential arc α'_i on F'_i with $\partial\alpha'_1 = \partial\alpha'_2 \subset \beta$. Moreover, if F'_i is nonplanar, choose α'_i to be nonseparating on F'_i. Then $\alpha' = \alpha'_1 \cup \alpha'_2$ is ∂-nonseparating, and intersects β essentially in two points, as required.

Isotope α' so that it intersects $\partial G'$ minimally. The geometric intersection number between α' and β is 1 or 2, so $\alpha' \cap \partial G' \neq \emptyset$. Since α' is ∂-nonseparating, by our assumption above it is parallel into F, so there is an annulus A with $\partial A = \alpha' \cup \alpha$, where $\alpha \subset F$. Isotope A rel α' so that it intersects $(\partial G') \times I$ minimally. Since G' is incompressible, $(\partial G') \times I$ consists of incompressible annuli in M, hence $A \cap ((\partial G') \times I)$ has no trivial circles. Since F and F' are also incompressible, one can show that $A \cap ((\partial G') \times I)$ has no trivial arcs on A either. Therefore $A \cap ((\partial G') \times I)$ consists of vertical arcs $(\alpha' \cap \partial G') \times I$. These arcs cut A into several squares $\alpha'_i \times I$, where each α'_i is the closure of a component of $\alpha' - \partial G'$. Choose i so that α'_i lies outside of G'. Let H be the component of $F' - \mathrm{Int} G'$ that contains α'_i. Then $G'' = G' \cup N(\alpha'_i)$ is a surface parallel into F, and $\chi(G'') = \chi(G') - 1$. The arc α'_i is essential on H, so the only case that some boundary component γ of G'' bounds a disk on F' is when H is an annulus, and γ is the boundary of the disk obtained by cutting H along α'_i. Since F and F' are incompressible and M is irreducible, both ends of the annulus $\gamma \times I \subset G'' \times I \subset M$ bound disks on $F \cup F'$, which together with $\gamma \times I$ bounds a 3-ball in M. It follows that $G' \cup H$ is parallel into F. Since $G' \cup H$ has the same Euler characteristic as G' but fewer number of boundary components, this contradicts the choice of G'. Therefore $\partial G''$ consists of essential curves on F'. Since F' is incompressible, G'' is also incompressible. Since $\chi(G'') < \chi(G')$, this again contradicts the choice of G'. □

Given a simple closed curve α and a proper arc γ on F', denote by $\tau_\alpha^n \gamma$ the curve obtained from γ by Dehn twist n times along α, and by $N(\tau_\alpha^n \gamma)$ a regular neighborhood of $\tau_\alpha^n \gamma$ on ∂M. Suppose T is a fixed torus boundary component of a 3-manifold M. Denote by $M(r)$ the manifold obtained by Dehn filling on T along a slope r on T, that is $M(r)$ is obtained by gluing a solid torus V to M along T so that the curve r on T bounds a meridian disk on V. Denote by $\Delta(r_1, r_2)$ the minimal geometric intersection number between two slopes r_1, r_2. The following two theorems will be used in the proof of Theorem 9, which proves Theorem 1 in the case that F' contains a ∂-nonseparating curve which is not parallel into F.

Lemma 7. ([Wu2], Theorem 1) *Let T be a torus component on the boundary of a 3-manifold M, and let S be an incompressible surface on $\partial M - T$. Suppose there is no incompressible annulus in M with one boundary component on each of S and T. If S is compressible in $M(r_1)$ and $M(r_2)$, then $\Delta(r_1, r_2) \leq 1$. In particular, S is incompressible in all but at most three $M(r)$.* □

Lemma 8. ([CGLS], Theorem 2.4.3) *Let T, S, M be as in Lemma 7, and assume further that M is irreducible. Suppose that there is an incompressible annulus A in M with one boundary component on S and the other a curve r_0 on T. Then either S is a torus and $M = S \times I$, or S remains incompressible in all $M(r)$ with $\Delta(r, r_0) > 1$.* □

Theorem 9. *Let α be a ∂-nonseparating curve on F' which is not parallel into F, and let γ be a proper arc on F' intersecting α in one point. Then $F_n = F \cup N(\tau_\alpha^n \gamma)$ is incompressible for all but at most three consecutive n's.*

Proof. Let K be the knot obtained by pushing α slightly into M. There is an embedded annulus A_0 in M with $\partial A_0 = \alpha \cup K$. Consider the manifold $M_K = M - \mathrm{Int}N(K)$, where $N(K)$ is a regular neighborhood of K in M. Let T be the torus $\partial N(K)$, and let (m, l) be the meridian-longitude pair on T such that $l = A_0 \cap T$. Denote by $M_K(p/q)$ the manifold obtained by Dehn filling on T along the slope $pm + ql$. The Dehn twist τ_α^{-n} on F' extends to a Dehn twist of M_K along the annulus $A = A_0 \cap M_K$, which sends the meridian slope m of T to the slope $m - nl$, so it extends to a homeomorphism $\varphi_n : M = M_K(1/0) \cong M_K(-1/n)$, which maps the curve $\tau_\alpha^n \gamma$ to the curve γ, and hence the surface F_n to the surface $F_0 = F \cup N(\gamma)$. It follows that φ_n is a homeomorphism of pairs

$$\varphi_n : (M, F_n) \to (M_K(-1/n), F_0).$$

Therefore to prove the theorem we need only show that for all but at most three consecutive integers n, the surface F_0 is incompressible in $M_K(-1/n)$.

CLAIM 1. *$T = \partial N(K)$ is incompressible in M_K, and M_K is irreducible.*

If D is a compressing disk of T in M_K, then ∂D must intersect the meridian m of K in one point, because otherwise after the trivial Dehn filling, $M = M_K(1/0)$ would contain a lens space or $S^2 \times S^1$ summand, contradicting the irreducibility of M. It follows that K, and hence α, bounds a disk in M. In this case α is parallel to a trivial curve on F, which contradicts the assumption that α is not parallel into F. Similarly, if M_K is reducible, then since M is irreducible, K is contained in a ball in M, so α would be null-homotopic. Using Dehn's Lemma, we see that α bounds a disk in M, hence is parallel to a trivial circle in F, contradicting the assumption that α is not parallel into F.

CLAIM 2. *F_0 is incompressible in M_K.*

Recall that A denotes the annulus $A_0 \cap M_K$. Since α intersects γ in a single point, $A \cap F_0$ is a single arc C on the boundary curve α of A. Let D be a compressing disk of F_0, chosen so that $|D \cap A|$, the number of components in $D \cap A$, is minimal. After disk swapping along disks on A bounded by innermost circles, we can assume that no component of $D \cap A$ is a trivial circle on A. Since T is incompressible by Claim 1, the annulus A is also incompressible, so $D \cap A$ contains no essential circle component on A either. Hence $D \cap A$ consists of arcs only. If some arc e of $D \cap A$ is parallel to a sub-arc on $C = A \cap F_0$, then after boundary compressing D along a disk Δ cut off by an outermost such arc we will get two disks D_1, D_2 with boundary on F_0, at least one of which has boundary an essential curve on F_0, hence is a compressing disk of F_0. Since $|D_i \cap A| < |D \cap A|$, this contradicts the minimality of $|D \cap A|$. Therefore, all arcs of $D \cap A$ are essential relative to C, in the sense that it is not parallel to an arc on C. See Figure 1(a). Notice that $|D \cap A| \neq 0$, otherwise D would be a compressing disk of F, contradicting the incompressibility of F.

Consider an outermost disk Δ on D, as shown in Figure 1(b). Then $\partial\Delta$ consists of two arcs e_1, e_2, where e_1 is an arc on A which is essential relative to C, and e_2 is an arc on F_0 with interior disjoint from C. Thus $e_2 \cap N(\gamma)$ consists of two arcs e_2', e_2''. Let t_1 be the subarc of C connecting the two ends of $e_2' \cup e_2''$ on C, and let t_2 be the subarc on $\partial N(\gamma)$ connecting the other two ends of $e_2' \cup e_2''$. Then $e_2' \cup t_1 \cup e_2'' \cup t_2$ bounds a disk Δ' on $N(\gamma)$. Now $A' = \Delta \cup \Delta'$ is an annulus in M, with one boundary component $e_1 \cup t_1$ an essential circle on A, which is parallel to α, and the other component $e_2 \cup t_2$ a curve on F. This contradicts the assumption that α is not parallel into F.

Figure 1

CLAIM 3. *There is no incompressible annulus P in M_K with one boundary component C_1 on F_0 and the other component C_2 a curve on T which is disjoint from $l = A \cap T$.*

The proof is similar to that of Claim 2. Choose P so that $|P \cap A|$ is minimal. Using the fact that P is incompressible, one can show as above that $P \cap A$ has no trivial circle component. Note that since C_2 is disjoint from l, $P \cap A$ has no arc component with endpoints on $l = A \cap T$. If $P \cap A$ had some essential circle component,choose such a component t which is closest to l on A. By cutting and pasting along the annulus on A bounded by $t \cup l$, one would get another incompressible annulus P' which has fewer intersection components with A. As in the proof of Claim 2 one can eliminate all arc components of $P \cap A$ which are inessential relative to $C = A \cap F_0$. Hence $P \cap A$ consists of arcs with ends on C and are essential relative to C, as shown in Figure 1(a). Also, since P is disjoint from l, $P \cap A$ are inessential arcs on P. Now one can use a disk Δ cut off by an outermost arc on P, proceed as in the proof of Claim 2 to get an annulus with one boundary on α and the other on F, and get a contradiction. Finally, if $P \cap A = \emptyset$ then P extends to an annulus with one boundary on α and the other on F, contradicting the assumption that α is not parallel into F. This completes the proof of Claim 3.

We now continue with the proof of Theorem 9. We have shown that F_0 is incompressible in M_K. If there is no essential annulus in M_K with one boundary component on each of F_0 and T, then by Lemma 7 we know that F_0 is incompressible in $M_K(r)$ for all but at most three slopes r with mutual intersection number 1. In particular, it is incompressible in $M_K(-1/n)$ for all but at most two consecutive n's, so the theorem follows. Now suppose there is an essential annulus P in M_K with one boundary component on F_0 and the other a curve r_0 on T. Since F_0 is not a closed surface, it is not a torus. Hence by Lemma 8, F_0 remains incompressible in $M_K(-1/n)$ unless $\Delta(-1/n, r_0) \leq 1$. By Claim 3, r_0 is not the longitude slope 0/1, therefore, $\Delta(-1/n, r_0) \leq 1$ holds for at most three consecutive integers n. This completes the proof of Theorem 9. $\square$

Proof of Theorem 1. By Theorem 4, Lemmas 5 and 6, and Theorem 9, we can now assume that F' is incompressible and is parallel into F. We want to show that either F is arc-extendible in F', or M is a product $F \times I$. As in the proof of Theorem 4, we may assume without loss of generality that M is a compression body, so all closed incompressible surfaces of M are boundary parallel. Let $\alpha_1, \ldots, \alpha_k$ be the boundary curves of F'. Let $F' \times I$ be a product in M such that $F' = F' \times 0$ and $F' \times 1 \subset F$. Write $\alpha_i^1 = \alpha_i \times 1$, which is a curve on F isotopic to α_i in M.

We have assumed above that F' is incompressible in M, so $\mathrm{Int}F \cup \mathrm{Int}F'$ is incompressible in M. Write $\widehat{F}_i = \mathrm{Int}F \cup \mathrm{Int}F' \cup \alpha_i$. If $\widehat{F}_i$ is incompressible for some i, then $F \cup N(\gamma)$ is incompressible for any essential arc γ in F' with endpoints on α_i, and we are done. (Such an arc exists because F' is not an annulus or disk.) So assume that $\widehat{F}_i$ is compressible for all i. By the Handle Addition Lemma (Lemma 2), after adding a 2-handle to M along α_i, the surface $\widehat{F}_i[\alpha_i]$ is incompressible, and $M[\alpha_i]$ is irreducible. Notice that since F' is incompressible, the curve $\alpha_i^1 = \alpha_i \times 1$ in F is essential on F. But after adding the 2-handle, α_i^1 bounds a disk in $M[\alpha_i]$, so it must also bound a disk on $\widehat{F}_i[\alpha_i]$ because $\widehat{F}_i[\alpha_i]$ is incompressible. By definition $\widehat{F}_i[\alpha_i]$ is obtained from $(\mathrm{Int}F \cup \mathrm{Int}F') - \mathrm{Int}N(\alpha_i)$ by capping off the two copies of α_i with disks, hence $\alpha_i^1 \cup \alpha_i$ bounds an annulus A_i on F_i. Denote by A_i' the annulus $\alpha_i \times I \subset F' \times I \subset M$. Then $T_i = A_i \cup A_i'$ is a torus in M. Since we have assumed above that M is a compression body, either T_i bounds a solid torus V_i, or it is parallel to some torus component of ∂M. However, since $M[\alpha_i]$ is irreducible, one can show as in the proof of Theorem 4 that V_i is a solid torus, and α_i is a longitude of V_i. This is true for all i. It is now easy to see that M is a product $F \times I$. $\square$

The following theorem supplements Theorem 1. It says that in most case there are extension arcs with endpoints on any prescribed boundary compponents of F'.

Theorem 10. *Let F, F', M be as in Theorem 1. Suppose M is not a product $F \times I$, and suppose F' is not parallel into F and is not a thrice punctured sphere. Then it contains an extension arc γ of F with endpoints on any prescribed components of $\partial F'$.*

Proof. If F' is nonplanar, then by the proof of Lemmas 5 and 6, there is a ∂-nonseparating circle α (denoted by C' there) on F' which is not parallel into F, and is actually nonseparating on F'. Hence given any boundary components ∂_1, ∂_2 of F', (possibly $\partial_1 = \partial_2$), there is an arc γ with endpoints on ∂_1 and ∂_2, intersecting α in one point. By Theorem 9, for all but at most three integers n, the arc $\gamma_n = \tau_\alpha^n \gamma$ is an extension arc of F.

Now suppose F' is planar with $|\partial F'| \geq 4$. First assume that ∂_1, ∂_2 are distinct boundary components of F'. By the proof of Lemmas 5 and 6, the curve α is a band sum of two boundary components of F'. From the proofs one can see that we can always choose α to be the band sum of ∂_1 and ∂_3, with $\partial_3 \neq \partial_1, \partial_2$. Hence there is an arc γ from ∂_1 to ∂_2 intersecting α in one point. We can then apply Theorem 9 to get an extension arc γ_n with one endpoint on each of ∂_1 and ∂_2.

We now proceed to find an extension arc in F' with boundary on the same component ∂_1 of $\partial F'$. By the proof of Lemmas 5 and 6, we can choose the curve α above to be the band sum of of ∂_2 and ∂_3, with $\partial_1 \neq \partial_2, \partial_3$. Recall that α is not parallel into F. Choose an arc γ as follows. Let ∂_2' be a curve on F' parallel to ∂_2, let γ' be an arc connecting ∂_2' to ∂_1 intersecting α in one point, and let Q be the sub-surface $N(\gamma' \cup \partial_2')$ of F'. Then γ is the closure of the arc component of $\partial Q \cap \mathrm{Int}F'$, that is, γ is the arc component of the frontier of Q in F', see Figure 2 below. Consider the surface $F_0 = F \cup N(\gamma)$, and observe that F_0 is isotopic to the surface $F \cup Q$. After Dehn twist along α, it is isotopic to the surface $F \cup N(\tau_\alpha^n \gamma)$; hence to show that all but at most three $\tau_\alpha^n \gamma$ are extension arcs of F in F', we need only show that $F \cup Q$ is incompressible after all but at most three Dehn twist along α. Since $F \cup Q$ intersects α

in a single arc, the argument in the proof of Theorem 9 is still valid, with the following easy modifications. We use the notations in that proof.

Figure 2

The proof of Claim 2 needs the following modifications. (i) The arc e_2 on the boundary of the outermost disk Δ may be on Q. In this case, notice that the other arc e_1 on ∂D is isotopic to an arc α_1 on α, and $e_2 \cup \alpha_1$ is isotopic in F' to the curve ∂_3, so the fact that $e_1 \cup e_2$ bounds a disk Δ would imply that ∂_3 bounds a disk. Since ∂_3 is also on ∂F, this contradicts the fact that F is incompressible and diskless. (ii) The compressing disk D of $F \cup Q$ could be disjoint from the annulus A. But since F is incompressible, this would imply that ∂D lies on Q, hence is isotopic to ∂_2, which would imply that ∂_2 bounds a disk, again contradicting the assumption that F is incompressible and diskless.

The proof of Claim 3 applies to show that the annulus P there can be modified to be disjoint from the annulus A. Then notice that the component of ∂P on $F \cup Q$ is either in F, or in Q and hence parallel to ∂_2. Since $\partial_2 \subset F$, in either case P can be extended to an annulus with one boundary component on α and the other on F, which contradicts the assumption that α is not parallel into F.

The rest part of the proof of Theorem 9 follows verbatim to show that $F \cup Q$ is incompressible after all but at most three Dehn twist along α. $\square$

Remark. Theorem 10 is not true if F' is a thrice punctured sphere.

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