INTRINSIC CHIRALITY OF GRAPHS IN 3-MANIFOLDS

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Abstract. The main result of this paper is that for every closed, connected, orientable, irreducible 3-manifold M, there is an integer n_M such that any abstract graph with no automorphism of order 2 which has a 3-connected minor whose genus is more than n_M has no achiral embedding in M. By contrast, the paper also proves that for every graph γ, there are infinitely many closed, connected, orientable, irreducible 3-manifolds M such that some embedding of γ in M is pointwise fixed by an orientation reversing involution of M.

1. Introduction

The study of chirality originally developed as a tool to help predict and explain molecular behavior. In particular, a molecule is said to be *chiral* if it can chemically interconvert with its mirror image at room temperature, and otherwise it is said to be *achiral*. Since small molecules are normally rigid, whether or not a small molecule is chiral can be determined from a geometric model. However, a molecule which is flexible or can rotate around specific bonds can be achiral even if a rigid model of it is geometrically distinct from its mirror image. The existence of such non-rigid molecules was the original motivation for studying chirality of spatial graphs from a topological perspective. However, the topological chirality of spatial graphs is interesting to consider whether or not they represent molecular structures.

In particular, we say that a graph embedded in S^3 is *achiral*, if there is an orientation reversing homeomorphism of S^3 taking the graph to itself. Otherwise, we say the embedded graph is *chiral*. We can think of knots with vertices as examples of graphs where some embeddings are chiral and others are not. By contrast, there are abstract graphs which have the property that no matter how they are embedded in S^3, they are topologically chiral. In this case, the graph is said to be *intrinsically chiral* in S^3. In chemical terms, a molecule would be intrinsically chiral if it and all of its topological stereoisomers are chiral. Molecular Möbius ladder with an odd number of rungs (at least three) were the first molecules that were shown to be intrinsically chiral [2]. More generally, the following theorem provides a method

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for showing that many graphs (molecular and otherwise) are intrinsically chiral in S^3.

Theorem 1. [3] *Every non-planar abstract graph γ with no automorphism of order 2 is intrinsically chiral in S^3.*

It makes sense to call such graphs *intrinsically* chiral because the chirality of such a graph depends only on the abstract graph and not on the embedding of the graph in S^3. However, we can define chirality for graphs embedded in any 3-manifold, and ask whether a graph which is intrinsically chiral in S^3 would be intrinsically chiral in a different 3-manifold. We prove the following theorem which shows that no graph can be intrinsically chiral in every 3-manifold.

Theorem 2. *For every graph γ, there are infinitely many closed, connected, orientable, irreducible 3-manifolds M such that some embedding of γ in M is pointwise fixed by an orientation reversing involution of M.*

The proof of this result can be thought of as a generalization of the fact that every planar graph has an embedding in S^3 which is pointwise fixed by a reflection of S^3. On the other hand, our main result is the following generalization of Theorem 1, which shows that for any "nice" 3-manifold M, any 3-connected abstract graph with large enough genus and no involution is intrinsically chiral in M.

Theorem 3. *For every closed, connected, orientable, irreducible 3-manifold M, there is an integer n_M such that any abstract graph with no automorphism of order 2 which has a 3-connected minor λ with $\text{genus}(\lambda) > n_M$ is intrinsically chiral in M.*

Note that by contrast with our result about embeddings of graphs in 3-manifolds without boundary, Ikeda [9] has shown in the theorem below that for "nice" 3-manifolds with aspherical boundary, any abstract graph with large enough genus which has a certain type of involution has an achiral hyperbolic embedding in the double of the manifold.

Ikeda's Theorem. [9] *Let M be a compact, connected, orientable, 3-manifold with non-empty aspherical boundary. Then there is an integer n_M such that for any abstract graph λ with $\text{genus}(\lambda) > n_M$ and no vertices of valence 1, any automorphism of order 2 of λ that does not restrict to an orientation preserving automorphism of a cycle in λ can be induced by an orientation reversing involution of some hyperbolic embedding of λ in the double of M.*

In Section 2, we prove Theorem 1. In Section 3, we determine the value of n_M for for a given manifold M. In Section 4, we prove Theorem 3 making use of a proposition, which we then prove in Section 5.

2. Achiral embeddings

The goal of this section is to prove Theorem 2. To that end, we prove the following proposition. Note that we use $\dim_{\mathbb{Z}}(H_1(M,\mathbb{Z}))$ to denote the dimension of the first $\mathbb{Z}$-homology group of M and $\dim_{\mathbb{Z}_2}(H_1(M,\mathbb{Z}_2))$ to denote the dimension of the first $\mathbb{Z}_2$-homology group of M.

Proposition 1. *Let S be a closed, orientable surface. Then for infinitely many closed, connected, orientable, irreducible 3-manifolds Q such that $\dim_{\mathbb{Z}_2}(H_1(Q,\mathbb{Z}_2)) = \text{genus}(S)$, there is an embedding of S in Q which is pointwise fixed by an orientation reversing involution of Q.*

The proof of Proposition 1 will make use of the idea of a *disk-busting* curve, which is a simple closed curve in a handlebody that intersects every essential, properly embedded disk in the handlebody. For example, a core of a solid torus is disk busting in the solid torus. For a genus 2 handlebody with fundamental group generated by a and b an example of a disk-busting curve is one that includes into the fundamental group of the handlebody as $abab^{-1}$ (see Figure 1). All handlebodies have disk-busting curves and Richard Strong [16] gives an algorithm to recognize them. For more on disk-busting curves see [16] or [8].

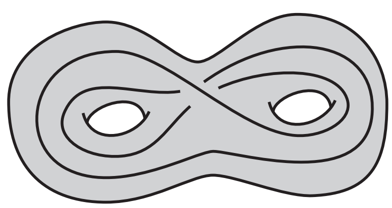

Figure 1. A disk-busting curve in a genus 2 handlebody.

Proof. Let g be the genus of a closed orientable surface S. Let M be the manifold obtained by gluing genus g handlebodies V_1 and V_2 together along S in such a way that there is an orientation reversing involution h interchanging V_1 and V_2 which pointwise fixes the surface S. Now M is a closed, connected, orientable 3-manifold M such that $\dim_{\mathbb{Z}_2}(H_1(M,\mathbb{Z}_2)) = \text{genus}(S)$. However, M is reducible.

In order to create an irreducible 3-manifold, we first remove neighborhoods N_1 and N_2 of identical disk busting curves in the interiors of the handlebodies V_1 and V_2 such that N_1 and N_2 are interchanged by the involution h. Note that since $\text{cl}(M - (S \cup N_1 \cup N_2))$ consists of two identical handlebodies from which neighborhoods of disk busting curves have been removed, the inclusion of S in each component of $\text{cl}(M - (S \cup N_1 \cup N_2))$ is incompressible. Now we sew in identical knot complements Q_1 and Q_2 along ∂N_1 and ∂N_2 respectively so that the Seifert surfaces of the Q_i are glued in where the meridians of the N_i were.

Let Q denote the 3-manifold obtained in this way. Then the restriction $h|\mathrm{cl}(M-(S\cup N_1\cup N_2))$ can be extended to an orientation reversing involution of Q that pointwise fixes S. Note that the components of $\mathrm{cl}(Q-(S\cup\partial Q_1\cup\partial Q_2))$ that contain S are homeomorphic to the corresponding components of $\mathrm{cl}(M-(S\cup\partial N_1\cup\partial N_2))$.

Claim 1: The surfaces S, ∂Q_1, and ∂Q_2 are each incompressible in Q

Proof of Claim 1: Assume one of S, ∂Q_1, or ∂Q_2 is compressible in Q. Then there is a compressing disk D for one of these surfaces that meets $S\cup\partial Q_1\cup\partial Q_2$ transversally in a minimal number of components. If the interior of D intersects one of the surfaces S, ∂Q_1, or ∂Q_2, then an innermost loop on D bounds a compressing disk for that surface whose interior is disjoint from the other surfaces. This implies that one of S, ∂Q_1, or ∂Q_2 is compressible in a component of $\mathrm{cl}(Q-(S\cup\partial Q_1\cup\partial Q_2))$. However, this is a contradiction because both Q_i and $\mathrm{cl}(V_i-N_i)$ have incompressible boundary.

Claim 2: Q is irreducible.

Proof of Claim: Let F be a sphere in Q and assume without loss of generality that F intersects $S\cup\partial Q_1\cup\partial Q_2$ transversally in a minimal number of components. If F intersects any of ∂Q_1, ∂Q_2, or S, then there is an innermost loop on F bounding a disk D that is a compressing disk for ∂Q_1, ∂Q_2, or S in $\mathrm{cl}(Q-(S\cup\partial Q_1\cup\partial Q_2))$. But this violates Claim 1. Thus F is contained in some Q_i or V_i-Q_i. However, both the knot complements Q_i and the V_i-Q_i are irreducible. Therefore Q is irreducible.

Now, recall that $\dim_{\mathbb{Z}_2}(H_1(M,\mathbb{Z}_2))=\mathrm{genus}(S)$. Also, it can be seen using a Meyer-Vietoris sequence that replacing the handlebodies N_1 and N_2 by the knot complements Q_1 and Q_2 does not change H_1, since a meridional disk of N_i is replaced by a Seifert surface in Q_i. Thus $\dim_{\mathbb{Z}_2}(H_1(Q,\mathbb{Z}_2))=\mathrm{genus}(S)$, and hence Q has the properties required by the proposition.

In order to prove that we can find infinitely many such 3-manifolds Q, we first show as follows that if Q_i is the complement of a connected sum of distinct knots $K_1\#K_2\#\dots\#K_n$, then Q contains at least $2n$ disjoint non-parallel incompressible tori. The first such torus is a boundary parallel torus in Q_i. The second such torus is a follow-swallow torus that swallows K_1 and follows the other $n-1$ tori. The third swallows $K_1\#K_2$ and follows the other $n-2$ and so on, each time swallowing one more knot and following one less until the n^{th} torus swallows all of the K_j with $j\neq n$. Since we have n such disjoint tori in each Q_i, this gives us $2n$ such disjoint tori in Q. These tori cannot be parallel to each other since they bound distinct knot complements in Q_i.

To see that these $2n$ tori are incompressible in Q, suppose there is a compressing disk for one of the tori that intersects $\partial Q_1\cup\partial Q_2\cup S$ transversally in a minimal number of components. An innermost loop on the disk would be a compressing disk for ∂Q_1, ∂Q_2 or S, again contradicting Claim 1.

Thus the torus would have to be compressible in one of the Q_i. But it is well known that follow-swallow tori are incompressible in knot complements, so the $2n$ tori must be incompressible in Q as well.

Thus Q contains $2n$ disjoint, non-parallel, incompressible tori. On the other hand, it follows from Kneser-Haken finiteness, that for a given compact, orientable manifold, such as Q, there is some finite constant t_1 such that Q cannot contain more than t_1 disjoint closed, non-parallel, incompressible surfaces (see, for example, Proposition 1.7 in [5]). Thus $t_1 > 2n$.

To get a manifold Q' which is not homeomorphic to Q, we replace each Q_i by the complement of a knot that is the connected sum of more than $\frac{1}{2}t_1$ knots. Now Q' will contain t_2 disjoint, non-parallel, incompressible tori with $t_2 > t_1$, and thus Q' is distinct from Q. By repeating this process, we can create an infinite sequence of such manifolds each containing more disjoint, non-parallel, incompressible tori than the previous manifold did. □

Since every graph embeds in a closed orientable surface, the following theorem is an immediate consequence of Proposition 1.

Theorem 2. *For every graph γ, there are infinitely many closed, connected, orientable, irreducible 3-manifolds M such that some embedding of γ in M is pointwise fixed by an orientation reversing involution of M.*

3. Determining the value of n_M for a given M

Let M be a closed, connected, orientable, irreducible 3-manifold. We will associate several constants with M that will help us determine the value of n_M. First, observe that we can apply the characteristic decomposition theorem of Jaco-Shalen [10] and Johannson [12] to M to find a unique minimal family Ω of disjoint incompressible tori such that the closures of the components obtained by splitting M along Ω are atoroidal or Seifert fibered. If $\Omega \neq \emptyset$ let $t = |\Omega|$, and otherwise let $t = 1$.

For each Seifert fibered component M_i, let g_i denote the genus of the base surface F_i, let b_i denote the number of boundary components of F_i, and let $w_i = \max\{b_i + 3g_i - 3, 1\}$. By using a standard pants decomposition argument (see, for example, [6]), we see that the surface F_i can contain at most w_i disjoint, non-parallel, non-boundary parallel, essential circles. This implies that M_i contains at most w_i disjoint, non-parallel, non-boundary parallel, incompressible vertical tori (see Figure 2). Let $w = \sum w_i$ taken over all Seifert fibered components M_i. Then M contains at most w disjoint, non-parallel, non-boundary parallel, incompressible tori which are vertical in some Seifert fibered component.

Now let T be an incompressible torus in M. Then T can be isotoped to be disjoint from the tori in the characteristic family Ω (see for example [4]). Thus without loss of generality, we assume that T is contained in either an atoroidal or a Seifert fibered component of $M - \Omega$. If T is in an atoroidal

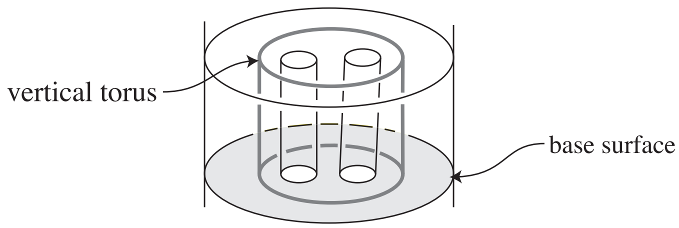

FIGURE 2. A vertical torus in a Seifert fibered component.

component, then T is parallel to a torus in Ω. Hence there are at most $t = |\Omega|$ disjoint, non-parallel, such tori in M. If T is in a Seifert fibered component, then by Waldhausen [18], T must be parallel to either a vertical or horizontal torus. As we saw above there are at most w disjoint, non-parallel, such tori which are vertical in some Seifert fibered component. If T is parallel to a horizontal torus in some Seifert fibered component, then M is Seifert fibered with base surface a torus. In this case, there is only one such torus, and $w = w_i = \max\{b_i + 3g_i - 3, 1\} = 1$. Thus there are at most w disjoint, non-parallel tori in M which are either parallel to either a vertical or horizontal torus in some Seifert fibered component. Hence altogether, M has at most $N_M = t + w$ disjoint, non-parallel incompressible tori. Note that since $w \geq 1$ and $t \geq 1$, we have $N_M \geq 2$.

Now for any closed, orientable, connected, irreducible 3-manifold M, we define $n_M = \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M$. We will refer to the constants n_M and N_M in the statements and proofs of Theorem 3 and Proposition 2.

4. Intrinsic Chirality

The goal of this section is to prove our main result. We begin with the following definition.

Definition 3. Let the *genus* of a graph γ be defined as the minimum value of

$$\frac{2 - \chi(S)}{2} = \frac{\dim_{\mathbb{Z}_2}(H_1(S, \mathbb{Z}_2))}{2}$$

over any surface S in which γ embeds.

It is worth pointing out that the genus of a non-orientable surface is not consistently defined in the literature. Some papers use our preferred definition, but others define the projective plane as having genus 1 instead of $\frac{1}{2}$.

In the following proposition and subsequent theorem, we refer to the values of N_M and n_M which were defined in Section 3.

Proposition 2. *Let γ be a 3-connected graph with genus at least 2, and let Γ be an embedding of γ in a closed, connected, orientable, irreducible 3-manifold M such that (M, Γ) has an orientation reversing homeomorphism fixing every vertex of Γ.*

Then there is an embedding Γ' of γ in a closed, connected, orientable 3-manifold M' such that (M', Γ') has an orientation reversing involution pointwise fixing Γ' and $\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M$

The point of this proposition is that if we have an embedding Γ of a graph γ in a manifold M such that (M, Γ) has an orientation reversing *homeomorphism*, then we can find another manifold M' and an embedding Γ' of γ in M' such that (M', Γ') has an orientation reversing *involution*. Furthermore, even though M' might be homologically more complicated than M, there is a bound on $\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2))$ which depends only on M and not on the graph γ or a particular embedding of γ in M.

Proposition 2 will be proved in the next section. We now prove Theorem 3 (restated below) by making use of Proposition 2 together with the following result of Kobayashi.

Kobayashi's Theorem. [13] *Let X be a closed, orientable, 3-manifold admitting an orientation reversing involution h. Then*

$$\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(X, \mathbb{Z}_2)) + \dim_{\mathbb{Z}}(H_1(X, \mathbb{Z})).$$

Theorem 3. *For every closed, connected, orientable, irreducible 3-manifold M, there is an integer n_M such that any abstract graph with no automorphism of order 2 which has a 3-connected minor λ with* $\text{genus}(\lambda) > n_M$ *is intrinsically chiral in M.*

Proof. Let γ be a graph with no automorphism of order 2. Suppose for the sake of contradiction that γ has an achiral embedding Γ in the manifold M. Let n_M and N_M be the constants associated with M that were defined in Section 3. Let λ be a 3-connected minor of γ. We will now show that λ satisfies the inequality

$$\text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M = n_M.$$

First observe that if $\text{genus}(\lambda) \leq 1$, then the above inequality is immediate since we saw in Section 3 that $N_M \geq 2$. Thus we assume that $\text{genus}(\lambda) \geq 2$.

Since Γ is an achiral embedding of γ in M, there is an orientation reversing homeomorphism f of the pair (M, Γ). Let φ denote the automorphism that f induces on the graph Γ. Now even though f does not necessarily have finite order, φ has finite order because Γ has a finite number of vertices. Hence we can express the order of φ as $2^r q$, where $r \geq 0$ and q is odd. Since f is orientation reversing and q is odd, $g = f^q$ is also an orientation reversing homeomorphism of (M, Γ). Now it follows that g induces the automorphism φ^q on Γ and $\text{order}(\varphi^q) = 2^r$. In particular, g^{2^r} fixes every vertex of Γ.

If $r \geq 1$, then $g^{2^{r-1}}$ would induce an order two automorphism on Γ. As we assumed that no such automorphism exists, we must have $r = 0$. Thus $g = g^{2^r}$ is an orientation reversing homeomorphism of (M, Γ) which fixes every vertex of Γ.

Now let λ be a 3-connected minor of the abstract graph γ. Then by deleting and/or contracting some edges of the embedding Γ of γ in M, we obtain an embedding Λ of λ in M. Furthermore, by composing the homeomorphism g with an isotopy in a neighborhood of each edge that was contracted, we obtain an orientation reversing homeomorphism of (M, Λ) which fixes every vertex of Λ.

Since λ is a 3-connected graph with genus$(\lambda) \geq 2$, we can now apply Proposition 2 to get an embedding Λ' of λ in a 3-manifold M' such that (M', Λ') has an orientation reversing involution h pointwise fixing Λ' and

$$\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M = \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M.$$

Let F be the component of the fixed point set fix(h) containing Λ', and let $x \in \Lambda'$. We put a metric d on M', and define a new metric d' by $d'(x, y) = d(x, y) + d(h(x), h(y))$. Then any neighborhood of x with respect to the metric d' will be setwise invariant under h. Now we can pick a neighborhood $N(x)$ with respect to d' such that $N(x)$ is homeomorphic to a ball. Then by Smith theory [15], since $h|N(x)$ is an orientation reversing involution of the ball $N(x)$, the fix point set of $h|N(x)$ is either a single point or a properly embedded disk. Since $N(x) \cap \Lambda$ contains more than one point, fix$(h|N(x))$ is a properly embedded disk, and hence F is a closed surface. Thus

$$\text{genus}(\lambda) \leq \text{genus}(F) = \frac{2 - \chi(F)}{2}$$

$$= \frac{\dim_{\mathbb{Z}_2}(H_1(F, \mathbb{Z}_2))}{2} \leq \frac{\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2))}{2}.$$

Hence we have the inequality

$$2\text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2).$$

Also, since h is an orientation reversing involution and M' is a closed orientable manifold, we can apply Kobayashi's Theorem [13] to obtain the inequality

$$\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) + \dim_{\mathbb{Z}}(H_1(M', \mathbb{Z})).$$

It follows that

$$\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2)) \leq 2\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)).$$

Combining the above inequalities, we now have

$$\text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)).$$

But M' was given by Proposition 2 such that

$$\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M.$$

Hence we obtain the required inequality

$$\text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M = n_M.$$

It now follows that if γ has a 3-connected minor whose genus is greater than n_M, then γ must be intrinsically chiral in M. □

5. Proof of Proposition 2

For the sake of completeness, we begin by proving the following elementary lemma that we will use in the proof of Proposition 2.

Lemma 1. *Let S be a punctured sphere with $n \geq 3$ boundary components $\{c_1, c_2 \ldots c_n\}$. Let $\{s_1, s_2 \ldots s_m\}$ be disjoint embedded loops on S each parallel to some c_i and let $\{\alpha_1, \ldots, \alpha_t\}$ be disjoint annuli on $S - \bigcup_{i=1}^m s_i$. Then the closure of some component of $S - (\bigcup_{i=1}^m s_i \cup \bigcup_{i=1}^t \alpha_i)$ is a sphere with at least three holes.*

Proof. First note that $\text{cl}(S - \bigcup_{i=1}^m s_i)$ consists of n annuli and a sphere with $n \geq 3$ holes. Call the closure of this sphere with holes F_1. Any annuli α_i that are not in F_1 are thrown out of the collection $\{\alpha_1, \ldots, \alpha_t\}$ to get a possibly smaller collection $\{\alpha_1, \ldots, \alpha_r\}$. Now it suffices to prove that the closure of some component of $F_1 - (\bigcup_{i=1}^m s_i \cup \bigcup_{i=1}^r \alpha_i)$ is a sphere with at least three holes.

Observe that the total number of boundary components of $\text{cl}(F_1 - \alpha_1)$ is $n+2$. There are three possible cases for these components. First, one component of $\text{cl}(F_1 - \alpha_1)$ could be a disk D_1. In this case, the total number of boundary components of $\text{cl}(F_1 - (\alpha_1 \cup D_1))$ is $n + 1 \geq 3$. Second, one component of $\text{cl}(F_1 - \alpha_1)$ could be an annulus A_1, in which case the total number of boundary components of $\text{cl}(F_1 - (\alpha_1 \cup A_1))$ is still $n \geq 3$. Finally, if neither component of $\text{cl}(F_1 - \alpha_1)$ is a disk or an annulus, then α_1 splits F into two punctured spheres each with at least three boundary components. Let F_2 denote the closure of one of these spheres with at least three holes.

We repeat the above paragraph inductively to conclude that the closure of some component of $S - (\bigcup_{i=1}^m s_i \cup \bigcup_{i=1}^t \alpha_i)$ is a sphere with at least three holes. □

We will also use the well known "Half Lives, Half Dies" Theorem, which we state below. See [7] or [5] for a proof of this theorem.

Theorem 4. (Half Lives, Half Dies) *Let M be a compact orientable 3-manifold. Then the following equation holds with any field coefficients*

$$\dim\left(Kernel(H_1(\partial M) \to H_1(M))\right) = \frac{1}{2} \dim H_1(\partial M).$$

Corollary 1. *Let M be a manifold which has a torus boundary component T. Then for any pair of generators a and b of $H_1(T, \mathbb{Z}_2)$, at least one of a and b is non-trivial in $H_1(M, \mathbb{Z}_2)$.*

Proof. Suppose for the sake of contradiction that the generators a and b are both trivial in $H_1(M, \mathbb{Z}_2)$. Attach handlebodies to all boundary components of M except T to form a new manifold J with a single boundary component. Then a and b are both trivial in $H_1(J, \mathbb{Z}_2)$. Since a and b generate the homology of the only boundary component of J, we see that $\dim(Kernel(H_1(\partial J, \mathbb{Z}_2)) \to H_1(J, \mathbb{Z}_2)) = \dim_{\mathbb{Z}_2}(H_1(\partial J, \mathbb{Z}_2)) = 2$. But this contradicts the Half Lives, Half Dies Theorem. Thus at least one of the generators of $H_1(T, \mathbb{Z}_2)$ must have been non-trivial in M. □

Note that it is tempting to assume that the Half Lives, Half Dies Theorem implies that one of a or b must be trivial in J. But this is not always true. In particular, let M denote the product of a torus and an interval. Then no single non-trivial curve in the boundary of M is in the kernel.

Now we are ready to prove Proposition 2. Recall that the definition of the constant n_M is given in Section 3.

Proposition 2. *Let γ be a 3-connected graph with genus at least 2, and let Γ be an embedding of γ in a closed, connected, orientable, irreducible 3-manifold M such that (M, Γ) has an orientation reversing homeomorphism g fixing every vertex of Γ.*

Then there is an embedding Γ' of γ in a closed, connected, orientable 3-manifold M' such that (M', Γ') has an orientation reversing involution pointwise fixing Γ' and $\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M$.

Proof. Let Λ denote either Γ or Γ with one edge deleted. Suppose that Λ is contained in a ball B in M. Since g fixes every vertex of Λ, without loss of generality, we can assume that Λ is pointwise fixed by g. Furthermore, since $g(B)$ is isotopic to B in M, we can assume that g leaves B setwise invariant. Let f be an embedding of (B, Λ) in S^3. Then $f \circ g \circ f^{-1}$ is an orientation reversing homeomorphism of $f(B)$ pointwise fixing $f(\Lambda)$. Now, $f \circ g \circ f^{-1}$ can be extended to an orientation reversing homeomorphism of S^3 pointwise fixing $f(\Lambda)$.

However, Jiang and Wang [11] showed that no graph containing one of the graphs $K_{3,3}$ or K_5 has an embedding in S^3 which is pointwise fixed by an orientation reversing homeomorphism of S^3. Thus Λ cannot contain $K_{3,3}$ or K_5, and hence is abstractly planar. But this implies that genus$(\Gamma) \leq 1$, which is contrary to hypothesis. Thus neither Γ nor Γ with an edge deleted can be contained in a ball in M. We will use this result later in the proof.

Since the remainder of the proof is quite lengthy, we break it into steps.

Step 1: We define a neighborhood $N(\Gamma)$.

Let V and E be the sets of vertices and edges of Γ respectively. For each vertex $v \in V$, define $N(v)$ to be a ball around v in M (i.e., a 0-handle containing v), and let $N(V)$ denote the union of the balls around the vertices. Also, for each edge $e \in E$, let $N(e) = D \times I$ be a solid tube around $\mathrm{cl}(e - N(V))$ in M (i.e., a 1-handle containing the portion of e outside of the 0-handles). Let $N(E)$ denote the union of the tubes around the edges. Then for each e in $N(E)$ the intersection $N(V) \cap N(e)$ follows the standard convention for attaching 1-handles to 0-handles. In other words, in $N(e)$ this intersection consists of the disks $D \times \{0\}$ and $D \times \{1\}$ and in $N(V)$ this intersection consists of these disks in the boundaries of the balls. Then $N(\Gamma) = N(E) \cup N(V)$ is a neighborhood of Γ.

For convenience we introduce the following terminology. For each vertex v, we let $\partial' N(v)$ denote the sphere with holes $\partial N(v) \cap \partial N(\Gamma)$, and for each edge e we let $\partial' N(e)$ denote the annulus $\partial N(e) \cap \partial N(\Gamma)$. Thus $\partial N(\Gamma) = \partial' N(E) \cup \partial' N(V)$.

Now since $g(\Gamma) = \Gamma$ fixing each vertex of Γ, we know that $g(N(\Gamma))$ is isotopic to $N(\Gamma)$ setwise fixing Γ and fixing each vertex. Thus we can modify g by an isotopy (and by an abuse of notation, still refer to the map as g) so that for each vertex v and edge e we have $g(N(v)) = N(v)$ and $g(N(e)) = N(e)$. Because this modification was by an isotopy, our new g is still orientation reversing.

Step 2: We split $\mathrm{cl}(M - N(\Gamma))$ along a family τ of JSJ tori and choose an invariant component X.

Since M is irreducible and we have assumed that Γ is not contained in a ball, $\mathrm{cl}(M - N(\Gamma))$ is irreducible. Thus we can apply the Characteristic Decomposition Theorem of Jaco-Shalen [10] and Johannson [12] to get a minimal family of incompressible tori τ for $\mathrm{cl}(M - N(\Gamma))$ such that each closed up component of $M - (N(\Gamma) \cup \tau)$ is either Seifert fibered or atoriodal. Since the characteristic family τ is unique up to isotopy, we can again modify g by an isotopy (and again by an abuse of notation still refer to the map as g) so that $g(\tau) = \tau$ and still have $g(N(v)) = N(v)$ and $g(N(e)) = N(e)$ for each vertex v and edge e. Let X be the closed up component of $M - (N(\Gamma) \cup \tau)$ containing $\partial N(\Gamma)$ (see for example Figure 3). Then $g(X) = X$.

Also, since Γ is 3-connected, $\mathrm{genus}(\partial N(\Gamma)) > 1$. Thus the component X is not Seifert fibered, and hence is atoroidal. Let P denote the set of torus boundary components of X together with the annuli that make up the components of $\partial' N(E)$. Since Γ is 3-connected, $\partial X - P = \partial N(\Gamma) - \partial' N(E) = \partial' N(V)$ is incompressible in $\mathrm{cl}(M - N(\Gamma))$. It follows that $\partial X - P$ is incompressible in X. Furthermore, X is irreducible since $\mathrm{cl}(M - N(\Gamma))$ is irreducible and X is a component of the JSJ decomposition of $\mathrm{cl}(M - N(\Gamma))$.

Step 3: We show that any sphere obtained by capping off an annulus in the JSJ decomposition of (X, P) bounds a ball in M intersecting at most one edge of $\Gamma - N(V)$.

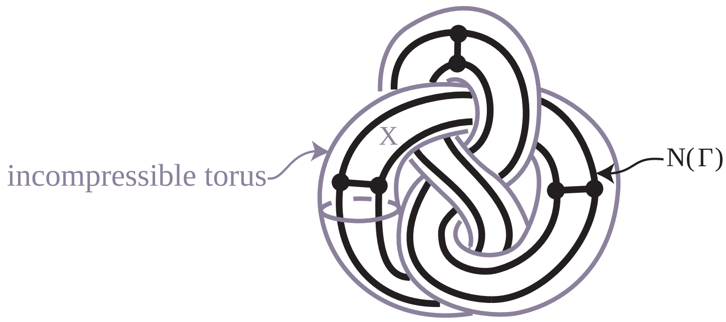

Figure 3. X is the closed up component of $M-(N(\Gamma)\cup\tau)$ between the grey incompressible torus and the black $\partial N(\Gamma)$.

We now apply the Characteristic Decomposition Theorem for Pared Manifolds [10, 12] to the pared manifold (X,P). Since X is atoroidal, this gives us a characteristic family σ of incompressible annuli in X with boundaries in $\partial X-P$ such that if W is the closure of any component of $X-\sigma$, then the pared manifold $(W, W\cap(P\cup\sigma))$ is either simple, Seifert fibered, or I - fibered (see [1] for the necessary definitions). Once again, since the characteristic family σ is unique up to isotopy, we can modify g by an isotopy (and again by an abuse of notation, still refer to the map as g) so that $g(\sigma)=\sigma$.

Let A be an annulus component of $P\cup\sigma$, and let S denote the sphere obtained by capping off A by a pair of disjoint disks D_1 and D_2 in $\partial N(v_1)$ and $\partial N(v_2)$, where v_1 and v_2 may or may not be distinct vertices. Suppose that each component of $M-S$ intersects more than one edge of $\Gamma-N(V)$. Then by removing the vertices v_1 and v_2 and the edges that contain them we would obtain two non-empty subgraphs (see Figure 4). But this contradicts our hypothesis that Γ is 3-connected. Thus one of the components of $M-S$ meets $\Gamma-N(V)$ in at most one edge of Γ.

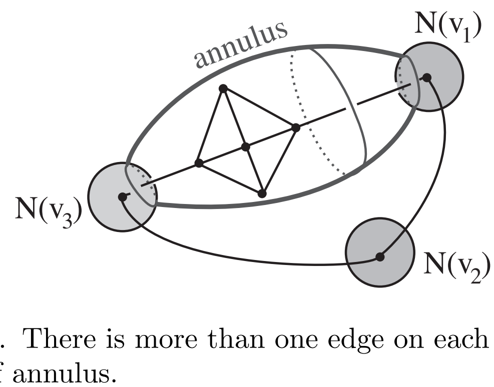

Figure 4. There is more than one edge on each side of this capped off annulus.

Now, since M is irreducible, one of the closed up components of $M - S$ is a ball B. However, we assumed at the beginning of our proof that neither Γ nor Γ with an edge removed can be contained in a ball in M. Thus B must be the closed up component of $M - S$ intersecting at most one edge of $\Gamma - N(V)$. Furthermore, since the annulus A is incompressible in X, if $v_1 \neq v_2$ then there is some edge e with vertices v_1 and v_2 such that B contains $\mathrm{cl}(e - (N(v_1) \cup N(v_2)))$. On the other hand, if $v_1 = v_2$, then since Γ is a graph B must be disjoint from $\Gamma - N(V)$.

Note that since a ball cannot contain an incompressible torus, no torus boundary component of X can occur in one of these balls. It follows that every torus component of ∂X must also be a component of ∂W.

Step 4: We define a collection of balls U_{e_1}, . . . , U_{e_n}, V_{F_1}, . . . , V_{F_m} in M such that every annulus in $P \cup \sigma$ is contained in some U_{e_i} if its boundaries are in distinct components of $\partial N(V)$, and in some V_{F_j} if its boundaries are in a single component of $\partial N(V)$.

Let A be an annulus in $P \cup \sigma$ with one boundary in $\partial N(v_1)$ and the other boundary in $\partial N(v_2)$ with $v_1 \neq v_2$. By capping off A with disks in $N(v_1)$ and $N(v_2)$ we obtain a sphere, which as we saw in Step 3, bounds a ball B that contains $\mathrm{cl}(e - (N(v_1) \cup N(v_2)))$ for some edge e in Γ. Now let $\mathcal{C}_e$ denote the collection of all annuli in $P \cup \sigma$ with one boundary in $\partial N(v_1)$ and the other boundary in $\partial N(v_2)$. By capping off the annuli in $\mathcal{C}_e$ with pairwise disjoint disks in $N(v_1)$ and $N(v_2)$, we obtain a collection of disjoint spheres which bound nested balls containing $\mathrm{cl}(e - (N(v_1) \cup N(v_2)))$. Let A_e denote the annulus in $\mathcal{C}_e$ which when capped off in this way is outermost with respect to this nesting. Observe that the boundaries of A_e also bound disks $D_1 \subseteq \partial N(v_1)$ and $D_2 \subseteq \partial N(v_2)$ which each meet Γ in a single point of e. Now the sphere $A_e \cup D_1 \cup D_2$ bounds a ball U_e in M which contains both $\mathrm{cl}(e - (N(v_1) \cup N(v_2)))$ and every annulus in $\mathcal{C}_e$.

We repeat the above paragraph for each annulus in $P \cup \sigma$ with boundaries in distinct components of $\partial N(V)$ to get a collection of pairwise disjoint balls $U_{e_1}, \ldots, U_{e_n}$ such that $U_{e_1} \cup \cdots \cup U_{e_n}$ contains both $\mathrm{cl}(\Gamma - N(V))$ and every annulus in $P \cup \sigma$ with boundaries in distinct components of $\partial N(V)$. Observe that for every edge e, the annulus $\partial' N(e)$ is in P. Thus every edge e is contained in some U_e.

Next we consider an annulus F in σ which has both boundaries in a single component of $\partial N(V)$. We saw in Step 3 that if we cap off F by any pair of disjoint disks in $N(V)$ we obtain a sphere which bounds a ball that is disjoint from $\Gamma - N(V)$. We can now cap off every such annulus by pairwise disjoint disks properly embedded in $N(V)$ such that the balls we get are disjoint from the set of vertices V. Since the annuli in σ are disjoint and the pairs of disks are disjoint, pairs of balls we obtain in this way will either be nested or disjoint. Thus for each annulus in σ with boundaries in a single component of $\partial N(V)$, we can choose an annulus $F \in \sigma$ which when capped off bounds an outermost ball V_F with respect to the nesting of the collection (see Figure 5).

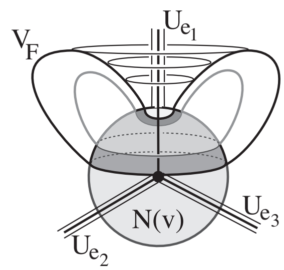

FIGURE 5. V_F is an outermost ball.

In this way we get a collection of disjoint balls $V_{F_1}, \ldots, V_{F_m}$ such that $V_{F_1} \cup \cdots \cup V_{F_m}$ contains every annulus in σ with boundaries in a single component of $\partial N(V)$. Furthermore, each such ball V_{F_i} is disjoint from $\Gamma - N(V)$, V, and from $U_{e_1} \cup \cdots \cup U_{e_n}$. Note that each $U_{e_j} \subseteq X$ and has only one boundary component; and each $V_{F_i} \cap X$ has only one boundary component. Thus both $X - U_{e_j}$ and $X - V_{F_i}$ have a single component. It follows that the manifold

$$W = \mathrm{cl}(X - (U_{e_1} \cup \cdots \cup U_{e_n} \cup V_{F_1} \cup \cdots \cup V_{F_m}))$$

is the closure of a single component of $X - \sigma$ (see Figure 6).

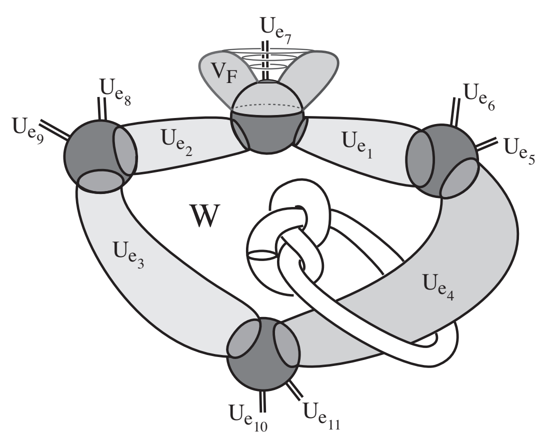

FIGURE 6. W is the closure of a single component of $X - \sigma$.

Step 5: We show that $g(W) = W$ and $(W, W \cap (P \cup \sigma))$ is simple as a pared manifold.

Recall that g fixes each vertex and leaves each edge setwise invariant. Also, $g(N(\Gamma)) = N(\Gamma)$, $g(P) = P$, and $g(\sigma) = \sigma$. Now, since the sets of balls $\{U_{e_1}, \ldots, U_{e_n}\}$ and $\{V_{F_1}, \ldots, V_{F_m}\}$ were chosen to be outermost, each of these sets is invariant under g. Furthermore, we know from Step 4 that each U_{e_j} intersects $\Gamma - N(V)$ only in e_j. Now since each edge is fixed by g, both A_{e_j} and U_{e_j} are setwise invariant under g; and since each vertex is fixed by g, each boundary component of A_{e_j} is also setwise invariant under g. Now let v be a vertex such that some F_i has both its boundary components in $\partial N(v)$. Since F_i is incompressible in X, each component of ∂F_i bounds a disk in $\partial N(v)$ which contains a boundary component of some U_{e_j}. Since the boundary components of each U_{e_j} are setwise invariant under g, it follows that F_i and its boundary components are also setwise invariant under g. ThusV_{F_i} is setwise invariant under g. Finally, since all of the U_{e_j} and V_{F_i} are invariant under g, we know that W must be setwise invariant under g as well.

To show that the pared manifold $(W, W \cap (P \cup \sigma))$ is simple, first recall that W is the closure of a single component of $X - \sigma$. Hence by JSJ for pared manifolds [10, 12], $(W, W \cap (P \cup \sigma))$ is either I-fibered, Seifert fibered, or simple as a pared manifold. We see that $(W, W \cap (P \cup \sigma))$ cannot be Seifert fibered or I-fibered as follows.

First observe that for every vertex v, there is some edge e such that the ball U_e meets $\partial N(v)$. It follows that ∂W meets every component of $\partial N(V)$. Furthermore, since every vertex v has valence at least three, $\partial' N(v)$ is a sphere with at least three holes. Also each U_{e_j} contains at most one boundary component of $\partial' N(v)$, and each V_{F_i} contains no boundary components of $\partial' N(v)$. Hence by Lemma 1, some component of $W \cap \partial N(v)$ is a sphere with at least three holes. It follows that the component of ∂W meeting $\partial N(\Gamma)$ has genus more than one, and thus the pared manifold $(W, W \cap (P \cup \sigma))$ cannot be Seifert fibered.

Next suppose for the sake of contradiction that the pared manifold $(W, W \cap (P \cup \sigma))$ is I-fibered. By definition of I-fibered for pared manifolds, this means that there is an I-bundle map of W over a base surface Y such that $W \cap (P \cup \sigma)$ is in the pre-image of ∂Y. It follows that Y must be homeomorphic to a component of $\partial' N(V)$. This means that the base surface Y must be a sphere with holes. Now since M is orientable, in fact W is a product $Y \times I$. Thus $W \cap (P \cup \sigma) = \partial Y \times I$, and $Y_0 = Y \times \{0\}$ and $Y_1 = Y \times \{1\}$ are components of $\partial' N(V) \cap W$. However, since ∂W meets every component of $\partial N(V)$, this means that Γ contains at most two vertices. But this contradicts our hypothesis that Γ is 3-connected. Therefore, the pared $(W, W \cap (P \cup \sigma))$ is not I-fibered, and since it is also not Seifert fibered, it must be simple.

Step 6: We prove that $g|W$ is isotopic to an orientation reversing involution h of $(W, W \cap (P \cup \sigma))$.

Now it follows from Thurston's Hyperbolization Theorem for Pared Manifolds [17] applied to the simple pared manifold $(W, W \cap (P \cup \sigma))$ that $W - (W \cap (P \cup \sigma))$ admits a finite volume complete hyperbolic metric with totally geodesic boundary. Let D denote the double of $W - (W \cap (P \cup \sigma))$ along its boundary. Then D is a finite volume hyperbolic manifold, and $g|W$ can be doubled to obtain an orientation reversing homeomorphism of D (which we still call g) taking each copy of $W - (W \cap (P \cup \sigma))$ to itself. Now by Mostow's Rigidity Theorem [14] applied to D, the homeomorphism $g : D \to D$ is homotopic to an orientation reversing finite order isometry $h : D \to D$ that restricts to an isometry of $W - (W \cap (P \cup \sigma))$. By removing horocyclic neighborhoods of the cusps of $W - (W \cap (P \cup \sigma))$, we obtain a copy of the pair $(W, W \cap (P \cup \sigma))$ which is contained in $W - (W \cap (P \cup \sigma))$ and is setwise invariant under h. We abuse notation and now consider h to be an orientation reversing finite order isometry of $(W, W \cap (P \cup \sigma))$ instead of this copy. Furthermore, h induces isometries on the collection of tori and annuli in $W \cap (P \cup \sigma)$ with respect to a flat metric. Furthermore, the sets $\partial' N(V) \cap W$, $\partial' N(E) \cap W$, and $\tau \cap W$ are each setwise invariant under h. Finally, it follows from Waldhausen's Isotopy Theorem [19] that h is isotopic to $g|W$ by an isotopy leaving $W \cap (P \cup \sigma)$ setwise invariant.

Now, recall that the boundary components of W consist of tori in τ, and the union of spheres with holes in $\partial' N(V)$ together with annuli in $P \cup \sigma$. Recall from Step 5 that g leaves setwise invariant each annulus $A_{e_j} \subseteq \partial U_{e_j}$ with boundaries in distinct components of $\partial N(V)$, each annulus $F_i \subseteq \partial V_{F_i}$ with both boundaries in a single component of $\partial N(V)$, each component of ∂A_{e_j}, and each component of ∂F_i. Since h is isotopic to $g|W$ by an isotopy leaving $W \cap (P \cup \sigma)$ setwise invariant, h leaves invariant the same sets as g. It follows that for each vertex v, we have $h(\partial' N(v)) \cap W) = \partial' N(v) \cap W$, and h takes each component of $W \cap \partial N(v)$ to itself, leaving each boundary component setwise invariant.

Since h has finite order, h restricts to a finite order homeomorphism of every component of $W \cap \partial N(V)$. We saw in Step 5 that for every vertex v, at least one component C_v of $W \cap \partial N(v)$ is a sphere with at least three holes. Since h restricts to a finite order homeomorphism of C_v taking each boundary component of C_v to itself, h must be a reflection of C_v which also reflects each component of ∂C_v. Now h^2 is a finite order, orientation preserving isometry of W that pointwise fixes the surface C_v. It follows that h^2 is the identity, and hence h is an involution of W.

Step 7: We extend h to an orientation reversing involution of $X \cup N(\Gamma)$ which pointwise fixes a new embedding Γ' of γ.

Observe that since every annulus in $P \cup \sigma$ is incompressible in W, no component of $W \cap \partial N(V)$ can be a disk. Thus every component of $W \cap \partial N(V)$ is either an annulus or a sphere with at least three holes. As we saw in Step 6, for each vertex v, h reflects some component C_v of $W \cap \partial N(v)$ which is a sphere with at least three holes, and h reflects every component of ∂C_v. Let b_0 denote some boundary component of C_v. Then b_0 is also a boundary component of either an annulus A_{e_j} or an annulus F_i. Since h reflects b_0, we know that h must also reflect the annulus A_{e_j} or F_i, whichever contains b_0 in its boundary. Since the boundaries of the annulus are not interchanged, h must also reflect each boundary component of A_{e_j} or F_i. Below we extend h to U_{e_j} or V_{F_i}.

First we consider the case where b_0 is in the boundary of an annulus $A_{e_j} \subseteq \partial U_{e_j}$. Let D_j and D'_j denote the disks in $\mathrm{cl}(\partial U_{e_j} - A_{e_j})$. Then D_j and D'_j each meet Γ in a single point of e_j. Since h reflects the annulus A_{e_j} together with each boundary component of A_{e_j}, we can extend h radially to the disks D_j and D'_j to get a reflection of the sphere $A_{e_j} \cup D_j \cup D'_j$ pointwise fixing a circle containing the points $D_j \cap e_j$ and $D'_j \cap e_j$. Recall that the sphere $A_{e_j} \cup D_j \cup D'_j$ bounds the ball U_{e_j} in M. Now, we can express U_{e_j} as a product $D_j \times I$ whose core $\overline{e_j}$ has endpoints $D_j \cap e_j$ and $D'_j \cap e_j$ (see Figure 7). Now we extend h from a reflection of the sphere $A_{e_j} \cup D_j \cup D'_j$ to a reflection of the product $D_j \times I$ which pointwise fixes the core $\overline{e_j}$.

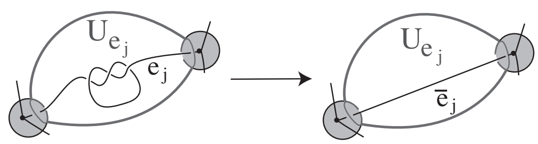

FIGURE 7. We can think of U_{e_j} as a product $D_j \times I$ with core $\overline{e_j}$.

Next we consider the case where b_0 is a boundary component of an annulus $F_i \subseteq \partial V_{F_i}$ which has both boundaries in a single $\partial N(v)$. Recall that $\mathrm{cl}(\partial V_{F_i} - F_i)$ consists of disks D_i and D'_i properly embedded in $N(v)$. Since h reflects the annulus F_i together with each of its boundary components, we can extend h radially to the disks D_i and D'_i to get a reflection of the sphere $F_i \cup D_i \cup D'_i$ pointwise fixing a circle containing all of the points of $\Gamma \cap D_i$ and $\Gamma \cap D'_i$. Recall that $V_{F_i} \cap \Gamma$ is a collection of one or more arcs. Thus we can extend h to a reflection of the ball V_{F_i} which pointwise fixing a disk containing $V_{F_i} \cap \Gamma$ (see Figure 8).

Now the extension of h reflects the sphere with holes C_v, and one of the balls U_{e_j} or V_{F_i} depending on whether b_0 is a boundary component of A_{e_j} or F_i, respectively. Next we let S_1 denote the union of C_v together with the annulus A_{e_j} or F_i glued along b_0. Now h reflects S_1 taking every boundary component of S_1 to itself, and hence reflecting every boundary component of S_1. Let b_1 be a boundary component of S_1. If b_1 is not the other boundary of the annulus A_{e_j} or F_i, then we repeat the above argument with b_1 in place of b_0. If b_1 is the other boundary of the annulus A_{e_j} or F_i, then b_1 is also a boundary of some other component S'_1 of $W \cap \partial N(V)$, as illustrated in Figure 9. In this case, since b_1 is reflected by h and every component of

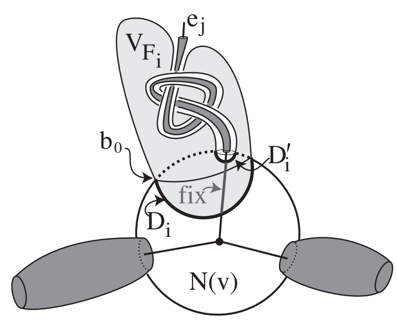

FIGURE 8. We extend h to a reflection of the ball V_{F_i} which pointwise fixes $\Gamma \cap V_{F_i}$.

S_1' is invariant under h, we know that h must reflect $S_2 = S_1 \cup S_1'$. Now let b_2 denote a boundary component of S_2, and repeat the above argument with b_2 in place of b_0.

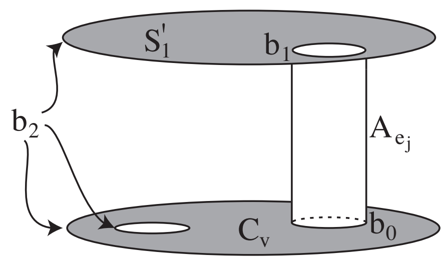

FIGURE 9. In this illustration, we have three choices for the boundary component b_2 of $S_2 = S_0 \cup A_{e_j} \cup S_1'$.

In general, for a given surface S_n obtained in this way, the surface S_{n+1} is the union of S_n together with either an annulus of the form A_{e_j} or F_i or a sphere with at least two holes contained in $W \cap \partial N(V)$. Furthermore, S_{n+1} is reflected by h. This process will only stop when the surface that we obtain has no boundary components. Since ∂W has only one component which intersects $\partial N(V)$, the closed surface that we obtain in this way must be ∂W. Thus we have extended h to an orientation reversing involution of each of the balls $U_{e_1}, \ldots, U_{e_n}, V_{F_1}, \ldots, V_{F_m}$.

Now let $N = N(V) - (V_{F_1} \cup \cdots \cup V_{F_m})$. Then N is a collection of disjoint balls (for example in Figure 8, $N(v) - V_{F_i}$ is two balls one of which contains v). Also, h is a reflection of each component of ∂N that fixes each point in $\partial N \cap \Gamma$. Now we extend h radially to a reflection of each ball of N in such a way that h pointwise fixes each component of $N \cap \Gamma$. Thus h is defined as a reflection of each component of $N(V)$ which pointwise fixes $N(V) \cap \Gamma$.

We have now extended h to an orientation reversing involution of the manifold

$$Y = W \cup V_{F_1} \cup \cdots \cup V_{F_m} \cup U_{e_1} \cup \cdots \cup U_{e_n} \cup N.$$

Recall from the end of Step 3 that ∂X and ∂W have the same collection of tori in their boundary components. Furthermore, we have filled in the boundary component of W meeting $\partial N(\Gamma)$ with a collection of balls in X and $N(\Gamma)$. Thus in fact $Y = X \cup N(\Gamma)$.

Finally, we define a new embedding Γ' of γ in $X \cup N(\Gamma)$ as follows. Let $\Gamma' \cap N(V) = \Gamma \cap N(V)$. Then for each edge e_j define an embedding of $e_j - N(V)$ in Γ' as the core $\overline{e_j}$ of $U_{e_j} = D_j \times I$, which we know is pointwise fixed by h according to the way we extended h to U_{e_j} (recall Figure 7).

Step 8: We prove that if an essential curve in a component of $\partial(X \cup N(\Gamma))$ compresses in M, then it compresses in $X \cup N(\Gamma)$.

Let $\{T_1, \ldots, T_r\}$ denote the set of boundary components of $X \cup N(\Gamma)$. These tori are contained in the characteristic family τ, and hence are incompressible in $\mathrm{cl}(M - N(\Gamma))$.

Suppose that an essential curve λ_i on some T_i compresses in M. Let D_i be a compressing disk for λ_i whose intersection with the set of tori $\{T_1, \ldots, T_r\}$ is minimal. Let $D = D_i$ if the interior of D_i is disjoint from T_i. Otherwise, there exists some D in the interior of D_i such that D_i is a compressing disk for T_i whose interior is disjoint from T_i. In either case, the intersection of D with $\{T_1, \ldots, T_r\}$ is minimal.

Suppose that D contains at least one curve of intersection in its interior. Hence there is an innermost disk Δ on D which is a compressing disk for some T_j with $j \neq i$. Since T_j compresses in M but is incompressible in $\mathrm{cl}(M - N(\Gamma))$, we know that Δ intersects Γ.

Since M is irreducible, any compressible torus is separating in M. Thus we can let X_j denote the closed up component of $M - T_j$ containing X and let V_j denote the closed up component of $M - T_j$ whose interior is disjoint from X. Now let S denote the region of D which is adjacent to the innermost disk Δ. Then $S \subseteq V_j$, since $\Delta \subseteq X_j$. Also, $\partial D \subseteq T_i \subseteq X \subseteq X_j$ implies that S is adjacent to another region of D which is contained in X_j. In particular, there must be another circle of intersection α of $D \cap T_j$ which bounds a disk $\overline{D} \subseteq D$ containing $\Delta \cup S$. We illustrate the abstract disk D and its intersections with T_j in Figure 10. The white regions in the figure are contained in V_j, and the grey regions are contained in X_j. Note we do not include any circles of intersection of D with any T_k with $k \neq j$.

Now, since the intersection of D with the tori $T_1,\ldots, T_r$ is minimal, all of the curves of intersection of $D \cap T_j$ must be essential on T_j. In particular, $\partial\Delta$ and α must both be essential on T_j. Since there cannot be two essential, disjoint, non-parallel curves on a torus, this means that α is parallel to $\partial\Delta$ on T_j. It follows that α must bound a disk $\overline{\Delta}$ which is parallel to Δ in M. In particular, since the interior of Δ is disjoint from $T_1,\ldots, T_r$, the interior of $\overline{\Delta}$ is as well. But now by replacing the disk $\overline{D}$ with the disk $\overline{\Delta}$ in the

FIGURE 10. A picture of the abstract disk D and its circles of intersection with T_j.

compressing disk D we obtain a new compressing disk D' which has fewer curves of intersection with $T_1,\dots,T_r$ than D has. From this contradiction we conclude that the interior of D must be disjoint from $T_1 \cup \cdots \cup T_r$, and hence $D \subseteq X \cup N(\Gamma)$.

If $\partial D = \lambda_i$, then λ_i compresses in $X \cup N(\Gamma)$ as required. Otherwise, the compression disk D was contained in the interior of the original disk D_1 and $\partial D \subseteq T_i$. In this case, since the intersection of D_1 with the set of tori $\{T_1, \dots, T_r\}$ was minimal, ∂D is essential in T_i. But now ∂D and λ_i are disjoint essential curves on T_i. Hence as we saw above, the disks D and D_1 must be parallel in M. Now, since $D \subseteq X \cup N(\Gamma)$, it follows that λ_i must compress in $X \cup N(\Gamma)$ as well.

Step 9: We fill each component T_i of $\partial(X \cup N(\Gamma))$ with a solid torus such that the manifold M' that we get satisfies the condition below.

Condition: *If T_i is compressible in M, then both generators of $H_1(T_i, \mathbb{Z}_2)$ are trivial in $H_1(M', \mathbb{Z}_2)$, and if T_i is incompressible in M then at least one generator of $H_1(T_i, \mathbb{Z}_2)$ is trivial in $H_1(M', \mathbb{Z}_2)$.*

Let T_i be a component of $\partial(X \cup N(\Gamma))$. By Corollary 1 there is a curve μ_i on T_i which is non-trivial in $H_1(X \cup N(\Gamma), \mathbb{Z}_2)$. Also, we know from Step 8 that if some essential curve λ_i on T_i compresses in M, then λ_i also compresses in $X \cup N(\Gamma)$. In particular, λ_i is not homologous in T_i to μ_i.

Now suppose that for some $j \neq i$, the involution h interchanges T_i and T_j. Since $h : X \cup N(\Gamma) \to X \cup N(\Gamma)$ is a homeomorphism and μ_i is a curve on T_i which is non-trivial in $H_1(X \cup N(\Gamma), \mathbb{Z}_2)$, we know that $h(\mu_i)$ is a curve on T_j which is also non-trivial in $H_1(X \cup N(\Gamma), \mathbb{Z}_2)$. Now we fill $X \cup N(\Gamma)$ along T_i by adding a solid torus V_i with its meridian attached to the non-trivial curve μ_i, and we fill along T_j by adding a solid torus V_j with its meridian attached to $h(\mu_i)$. Then we extend the involution h radially on $V_i \cup V_j$ (abusing notation and still calling the involution h). We repeat this process for every component of $X \cup N(\Gamma)$ which is not setwise invariant under h. As a result, for every T_i along which we have glued a solid torus V_i, the curve μ_i on T_i is trivial in $H_1(X \cup N(\Gamma) \cup V_i, \mathbb{Z}_2)$. Furthermore, if T_i is compressible in M, then there is an essential curve λ_i on T_i which compresses in $X \cup N(\Gamma)$. Now, λ_i is not homologous to μ_i in $H_1(T_i, \mathbb{Z}_2)$, and together they generate $H_1(T_i, \mathbb{Z}_2)$. Furthermore, both λ_i and μ_i are trivial in $H_1(X \cup N(\Gamma) \cup V_i, \mathbb{Z}_2)$

Let Z be the manifold that we have obtained by filling all of the boundary components of $X \cup N(\Gamma)$ which are not setwise fixed by h, and let T_i be a component of ∂Z. Recall from Step 7 that Γ' is an embedding of γ in $X \cup N(\Gamma)$ which is pointwise fixed by h. Now $h : Z \to Z$ is an orientation reversing involution pointwise fixing Γ', and T_i is setwise invariant under h. Since $h|T_i$ is an order 2 isometry of a torus, $h|T_i$ is either a reflection pointwise fixing two parallel circles on T_i, a rotation pointwise fixing four points of T_i, or a rotation fixing no points of T_i. In each of these cases, there is a pair of generators of $H_1(T_i, \mathbb{Z}_2)$ each of which is homologous to its image under h. It follows that for any given generator a_i of $H_1(T_i, \mathbb{Z}_2)$ (which may or may not be homologous to $h(a_i)$ in $H_1(T_i, \mathbb{Z}_2)$), there is a curve b_i on T_i such that $\langle a_i, b_i \rangle = H_1(T_i, \mathbb{Z}_2)$ and $h(b_i)$ is homologous to b_i in $H_1(T_i, \mathbb{Z}_2)$.

Now suppose that some essential curve λ_i on T_i compresses in M. Then by Step 8, λ_i also compresses in $X \cup N(\Gamma)$, and hence in Z. We can now pick a curve b_i on T_i such that $\langle \lambda_i, b_i \rangle = H_1(T_i, \mathbb{Z}_2)$ and $h(b_i)$ is homologous to b_i in $H_1(T_i, \mathbb{Z}_2)$. Since λ_i is null homologous in Z, by Corollary 1, b_i is non-trivial in $H_1(Z, \mathbb{Z}_2)$. Now we fill Z along T_i by adding a solid torus V_i with its meridian attached to the non-trivial curve b_i. Since $h(b_i)$ is homologous to b_i in $H_1(T_i, \mathbb{Z}_2)$, we can extend h radially to the solid torus V_i. Then $h : Z \cup V_i \to Z \cup V_i$ is an orientation reversing involution, and both λ_i and b_i are trivial in $H_1(Z \cup V_i, \mathbb{Z}_2)$.

Now suppose that some T_i is incompressible in M. As we saw above, there is a pair of generators of $H_1(T_i, \mathbb{Z}_2)$ each of which is homologous in T_i to its image under h. By Corollary 1, at most one of these generators is null homologous in Z. So there is some curve b_i on T_i which is non-trivial in $H_1(Z, \mathbb{Z}_2)$ and homologous to $h(b_i)$ on T_i. Now fill T_i by adding a solid torus V_i with its meridian attached to the curve b_i and extend h to V_i. Then $h : Z \cup V_i \to Z \cup V_i$ is again an orientation reversing involution, and b_i is trivial in $H_1(Z \cup V_i, \mathbb{Z}_2)$.

In this way, we glue a solid torus to each of the T_i in $\partial(X \cup N(\Gamma))$ to obtain a closed manifold M' satisfying the required condition. Since $\Gamma' \subseteq X \cup N(\Gamma)$, this gives us an embedding Γ' of γ in M'. Furthermore, we have extended h to an orientation reversing involution of (M', Γ') which pointwise fixes Γ'.

Step 10: We prove that there are at most N_M tori in $\partial(X \cup N(\Gamma))$ which are incompressible in M.

First suppose that some pair of distinct components T_i and T_j of $\partial(X \cup N(\Gamma))$ are parallel in M. Then T_i and T_j co-bound a region R in M which is homeomorphic to a product of a torus and an interval. However, since T_i and T_j are tori in the characteristic family for $\mathrm{cl}(M - N(\Gamma))$, they cannot be parallel in $\mathrm{cl}(M - N(\Gamma))$. Thus R intersects Γ. But since $\partial R = T_i \cup T_j$ and Γ is disjoint from $T_i \cup T_j$, this implies that $\Gamma \subseteq R$.

Suppose that $X \cup N(\Gamma)$ has a boundary component T_k which is distinct from T_i and T_j. Since $X \cup N(\Gamma)$ is a connected set which contains Γ and has T_i, T_j, and T_k among its boundary components, T_k must be contained in R. But since $R \cong T \times I$, either T_k is parallel in M to both T_i and T_j or T_k bounds a solid torus $V \subseteq R$. The former would imply that T_k is parallel to one of T_i or T_j in $\mathrm{cl}(M - N(\Gamma))$, which is impossible since all three are in the characteristic family for $\mathrm{cl}(M - N(\Gamma))$. However, the latter would imply that $N(\Gamma) \subseteq V$ because otherwise T_k would be compressible in $\mathrm{cl}(M - N(\Gamma))$. But, this is impossible since $\partial N(\Gamma)$, T_k, T_i, and T_j are all boundary components of X. Hence we must have $\partial(X \cup N(\Gamma)) = T_i \cup T_j$. Since we saw in Section 3 that $N_M \geq 2$, it now follows that $\partial(X \cup N(\Gamma))$ has at most N_M components as required. Thus we can assume that no pair of distinct components of $\partial(X \cup N(\Gamma))$ are parallel in M.

Let T_i be a component of $\partial(X \cup N(\Gamma))$ which is incompressible in M. Since Ω is the characteristic family of tori for M, we know that T_i can be isotoped to be disjoint from Ω (see for example [4]). Thus, without loss of generality, we can assume that T_i is contained in a closed up component of $M - \Omega$ which is either atoroidal or Seifert fibered. If T_i is in an atoroidal component, then T_i is parallel to a torus in Ω. If T_i is in a Seifert fibered component, then it follows from Waldhausen [18] that T_i is parallel to either a torus in Ω or a vertical or horizontal torus of the fibration.

Since no pair of distinct components of $\partial(X \cup N(\Gamma))$ are parallel in M, there are at most $t = |\Omega|$ incompressible tori in $\partial(X \cup N(\Gamma))$ that are parallel to a torus in Ω, and at most w (see Section 3 for the definition of w) incompressible tori in $\partial(X \cup N(\Gamma))$ that are parallel to a vertical or horizontal torus in some Seifert fibered closed up component of $M - \Omega$. Hence there are at most $N_M = t + w$ tori in $\partial(X \cup N(\Gamma))$ that are incompressible in M.

Step 11: We prove the inequality $\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M$.

Recall that M' is obtained from $X \cup N(\Gamma)$ by adding a collection of solid tori $V_1, \dots, V_r$ along the tori $T_1, \dots, T_r$.

Let $[\beta]_{M'}$ be a non-trivial element of $H_1(M', \mathbb{Z}_2)$. Now for each solid torus V_i, let C_i denote its core. Then by general position we can choose a representative curve for $[\beta]_{M'}$ that is disjoint from $C_1 \cup \dots \cup C_r$. Hence we can assume that β is disjoint from $V_1, \dots, V_r$. Now since $\beta \subseteq X \cup N(\Gamma) \subseteq M$, we can also consider the element $[\beta]_M \in H_1(M, \mathbb{Z}_2)$.

Suppose that $[\beta]_M$ is trivial in $H_1(M, \mathbb{Z}_2)$. It follows that β is homologous in $X \cup N(\Gamma)$ to a collection of curves on $T_1 \cup \dots \cup T_r$ which are trivial in M but non-trivial in M'. Recall from the condition in Step 9 that if T_i is compressible in M, then both generators of $H_1(T_i, \mathbb{Z}_2)$ are trivial in $H_1(M', \mathbb{Z}_2)$, and if T_i is incompressible in M then at least one generator of $H_1(T_i, \mathbb{Z}_2)$ is trivial in $H_1(M', \mathbb{Z}_2)$. For each T_i which is incompressible in M, if there is a generator of $H_1(T_i; Z_2)$ that is not trivial in $H_1(M', \mathbb{Z}_2)$ we denote it by β_i. Then every curve on T_i is either trivial in $H_1(M', \mathbb{Z}_2)$ or homologous to β_i in $H_1(M', \mathbb{Z}_2)$.

Returning now to the curve β which is homologous in M' to a sum of curves on $T_1 \cup \cdots \cup T_r$. Since for any T_i which is compressible in M both generators of $H_1(T_i, \mathbb{Z}_2)$ are trivial in $H_1(M', \mathbb{Z}_2)$, it now follows that β is homologous in $H_1(M', \mathbb{Z}_2)$ to a sum of β_i's on T_i's that are incompressible in M. But by Step 10, there are at most N_M such tori. Hence there are at most N_M such β_i that are not trivial in $H_1(M', \mathbb{Z}_2)$. It follows that these N_M curves generate every non-trivial $[\beta]_{M'}$ which is trivial in M. This give us the required inequality:

$$\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M = n_M.$$

Hence the proposition follows. $\square$

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