

Intrinsic linking and knotting of graphs in arbitrary 3–manifolds

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We prove that a graph is intrinsically linked in an arbitrary 3–manifold M if and only if it is intrinsically linked in S^3 . Also, assuming the Poincaré Conjecture, we prove that a graph is intrinsically knotted in M if and only if it is intrinsically knotted in S^3 .

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1 Introduction

The study of intrinsic linking and knotting began in 1983 when Conway and Gordon [1] showed that every embedding of K_6 (the complete graph on six vertices) in S^3 contains a non-trivial link, and every embedding of K_7 in S^3 contains a non-trivial knot. Since the existence of such a non-trivial link or knot depends only on the graph and not on the particular embedding of the graph in S^3 , we say that K_6 is *intrinsically linked* and K_7 is *intrinsically knotted*.

At roughly the same time as Conway and Gordon’s result, Sachs [12, 11] independently proved that K_6 and $K_{3,3,1}$ are intrinsically linked, and used these two results to prove that any graph with a minor in the *Petersen family* (Figure 1) is intrinsically linked. Conversely, Sachs conjectured that any graph which is intrinsically linked contains a minor in the Petersen family. In 1995, Robertson, Seymour and Thomas [10] proved Sachs’ conjecture, and thus completely classified intrinsically linked graphs.

Examples of intrinsically knotted graphs other than K_7 are now known, see Foisy [2], Kohara and Suzuki [3] and Shimabara [13]. Furthermore, a result of Robertson and Seymour [9] implies that there are only finitely many intrinsically knotted graphs that are minor-minimal with respect to intrinsic knottedness. However, as of yet, intrinsically knotted graphs have not been classified.

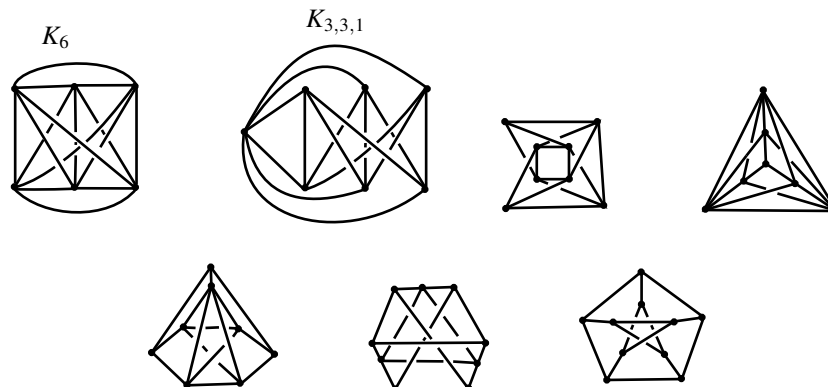


Figure 1: The Petersen family of graphs

In this paper we consider the properties of intrinsic linking and knotting in *arbitrary* 3-manifolds. We show that these properties are truly *intrinsic* to a graph in the sense that they do not depend on either the ambient 3-manifold or the particular embedding of the graph in the 3-manifold. Our proof in the case of intrinsic knotting assumes the Poincaré Conjecture.

We will use the following terminology. By a *graph* we shall mean a finite graph, possibly with loops and repeated edges. Manifolds may have boundary and do not have to be compact. All spaces are piecewise linear; in particular, we assume that the image of an *embedding* of a graph in a 3-manifold is a piecewise linear subset of the 3-manifold. An embedding of a graph G in a 3-manifold M is *unknotted* if every circuit in G bounds a disk in M ; otherwise, the embedding is *knotted*. An embedding of a graph G in a 3-manifold M is *unlinked* if it is unknotted and every pair of disjoint circuits in G bounds disjoint disks in M ; otherwise, the embedding is *linked*. A graph is *intrinsically linked* in M if every embedding of the graph in M is linked; and a graph is *intrinsically knotted* in M if every embedding of the graph in M is knotted. (So by definition an intrinsically knotted graph must be intrinsically linked, but not vice-versa.)

The main results of this paper are that a graph is intrinsically linked in an arbitrary 3-manifold if and only if it is intrinsically linked in S^3 ([Theorem 1](#)); and (assuming the Poincaré Conjecture) that a graph is intrinsically knotted in an arbitrary 3-manifold if and only if it is intrinsically knotted in S^3 ([Theorem 2](#)). We use Robertson, Seymour, and Thomas' classification of intrinsically linked graphs in S^3 for our proof of [Theorem 1](#). However, because there is no analogous classification of intrinsically knotted graphs in S^3 , we need to take a different approach to prove [Theorem 2](#). In particular, the proof of [Theorem 2](#) uses [Proposition 2](#) (every compact subset of a simply connected 3-manifold is homeomorphic to a subset of S^3), whose proof in turn relies on the Poincaré Conjecture.

Our assumption of the Poincaré Conjecture seems reasonable, because Perelman [7, 8] has announced a proof of Thurston's Geometrization Conjecture, which implies the Poincaré Conjecture [4]. (See also Morgan and Tian [5].)

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2 Intrinsically linked graphs

In this section, we prove that intrinsic linking is independent of the 3-manifold in which a graph is embedded. We begin by showing (Lemma 1) that any unlinked embedding of a graph G in a 3-manifold lifts to an unlinked embedding of G in the universal cover. In the universal cover, the linking number can be used to analyze intrinsic linking (Lemma 2), as in the proofs of Conway and Gordon [1] and Sachs [12, 11]. After we've shown that K_6 and $K_{3,3,1}$ are intrinsically linked in any 3-manifold (Proposition 1), we use the classification of intrinsically linked graphs in S^3 , Robertson, Seymour, and Thomas [10], to conclude that any graph that is intrinsically linked in S^3 is intrinsically linked in every 3-manifold (Theorem 1).

We call a circuit of length 3 in a graph a *triangle* and a circuit of length 4 a *square*.

Lemma 1 *Any unlinked embedding of a graph G in a 3-manifold M lifts to an unlinked embedding of G in the universal cover \tilde{M} .*

Proof Let $f: G \rightarrow M$ be an unlinked embedding. $\pi_1(G)$ is generated by the circuits of G (attached to a basepoint). Since $f(G)$ is unknotted, every cycle in $f(G)$ bounds a disk in M . So $f_*(\pi_1(G))$ is trivial in $\pi_1(M)$.

Thus, an unlinked embedding of G into M lifts to an embedding of G in the universal cover \tilde{M} . Since the embedding into M is unlinked, cycles of G bound disks in M and pairs of disjoint cycles of G bound disjoint disks in M . All of these disks in M lift to disks in \tilde{M} , so the embedding of the graph in \tilde{M} is also unlinked. \square

Recall that if M is a 3-manifold with $H_1(M) = 0$, then disjoint oriented loops J and K in M have a well-defined linking number $\text{lk}(J, K)$, which is the algebraic intersection number of J with any oriented surface bounded by K . Also, the linking number is symmetric: $\text{lk}(J, K) = \text{lk}(K, J)$.

It will be convenient to have a notation for the linking number modulo 2: Define $\omega(J, K) = \text{lk}(J, K) \bmod 2$. Notice that $\omega(J, K)$ is defined for a pair of *unoriented* loops. Since linking number is symmetric, so is $\omega(J, K)$. If J_1, \dots, J_n are loops in an embedded graph such that in the list J_1, \dots, J_n every edge appears an even number of times, and if K is another loop, disjoint from the J_i , then $\sum \omega(J_i, K) = 0 \bmod 2$.

If G is a graph embedded in a simply connected 3-manifold, let

$$\omega(G) = \sum \omega(J, K) \bmod 2,$$

where the sum is taken over all *unordered* pairs (J, K) of disjoint circuits in G . Notice that if $\omega(G) \neq 0$, then the embedding is linked (but the converse is not true).

Lemma 2 *Let \tilde{M} be a simply connected 3-manifold, and let H be an embedding of K_6 or $K_{3,3,1}$ in \tilde{M} . Let e be an edge of H , and let e' be an arc in \tilde{M} with the same endpoints as e , but otherwise disjoint from H . Let H' be the graph $(H - e) \cup e'$. Then $\omega(H') = \omega(H)$.*

Proof Let $D = e \cup e'$.

First consider the case that H is an embedding of K_6 . We will count how many terms in the sum defining $\omega(H)$ change when e is replaced by e' . Let K_1, K_2, K_3 and K_4 be the four triangles in H disjoint from e (hence also disjoint from e' in H'), and for each i let J_i be the triangle complementary to K_i . The J_i all contain e . For each i , let $J'_i = (J_i - e) \cup e'$, and notice that

$$(1) \quad \omega(J'_i, K_i) = \omega(J_i, K_i) + \omega(D, K_i) \bmod 2.$$

Because each edge appears twice in the list K_1, K_2, K_3, K_4 , we have $\omega(K_1, D) + \omega(K_2, D) + \omega(K_3, D) + \omega(K_4, D) = 0 \bmod 2$. Thus, $\omega(K_i, D)$ is nonzero for an even number of i . It follows from Equation 1 that there are an even number of i such that $\omega(J'_i, K_i) \neq \omega(J_i, K_i)$. Thus, $\sum_{i=1}^4 \omega(J'_i, K_i) = \sum_{i=1}^4 \omega(J_i, K_i) \bmod 2$, and

$$\begin{aligned} \omega(H') &= \sum_{\substack{J, K \subseteq H' \\ \exists e' \notin J, K}} \omega(J, K) + \sum_{i=1}^4 \omega(J'_i, K_i) \bmod 2 \\ &= \sum_{\substack{J, K \subseteq H \\ \exists e \notin J, K}} \omega(J, K) + \sum_{i=1}^4 \omega(J_i, K_i) \bmod 2 \\ &= \omega(H) \end{aligned}$$

Next consider the case that H is an embedding of $K_{3,3,1}$. Let x be the vertex of valence six in H (and in H').

Case 1 e contains x . Then e is not in any square in H that has a complementary disjoint triangle. Let K_1, K_2 and K_3 be the three squares in H disjoint from e , and let J_1, J_2 and J_3 be the corresponding complementary triangles, all of which contain e . As in the K_6 case, let $J'_i = (J_i - e) \cup e'$ for each i ; again we have Equation 1. Every edge in the list K_1, K_2, K_3 appears exactly twice, so $\omega(K_1, D) + \omega(K_2, D) + \omega(K_3, D) = 0 \pmod 2$. Thus, $\omega(K_i, D)$ is nonzero for an even number of i ; and for an even number of i , $\omega(J'_i, K_i) \neq \omega(J_i, K_i)$. The other pairs of circuits contributing to $\omega(H)$ do not involve e . As in the K_6 case, it follows that $\omega(H') = \omega(H)$.

Case 2 e doesn't contain x . Let J_0 be the triangle containing e , and let K_0 be the complementary square. Let J_1 through J_4 be the four squares that contain e , but not x (so that they have complementary triangles); and let K_1 through K_4 be the complementary triangles. With J'_i defined as in the other cases, we again have Equation 1. Every edge appears an even number of times in the list K_0, K_1, K_2, K_3, K_4 , so $\sum_{i=0}^4 \omega(K_i, D) = 0 \pmod 2$, and $\omega(K_i, D) \neq 0$ for an even number of i . As in the other cases, it follows that for an even number of i , $\omega(J'_i, K_i) \neq \omega(J_i, K_i)$; and an even number of the terms in the sum defining $\omega(H)$ change when e is replaced by e' ; and $\omega(H') = \omega(H)$. □

Proposition 1 K_6 and $K_{3,3,1}$ are intrinsically linked in any 3-manifold M .

Proof Let G be either K_6 or $K_{3,3,1}$, and let $f: G \rightarrow M$ be an embedding. Suppose for the sake of contradiction that $f(G)$ is unlinked. Let \tilde{M} be the universal cover of M . By Lemma 1, f lifts to an unlinked embedding $\tilde{f}: G \rightarrow \tilde{M}$.

Let $\tilde{G} = \tilde{f}(G) \subseteq \tilde{M}$, and let \tilde{H} be a copy of G embedded in a ball in \tilde{M} . Isotope \tilde{G} so that \tilde{H} and \tilde{G} have the same vertices, but do not otherwise intersect. Then \tilde{G} can be transformed into \tilde{H} by changing one edge at a time – replace an edge of \tilde{G} by the corresponding edge of \tilde{H} , once for every edge. By repeated applications of Lemma 2, $\omega(\tilde{G}) = \omega(\tilde{H})$. Since \tilde{H} is inside a ball in \tilde{M} , Conway and Gordon's proof [1], and Sachs' proof [12, 11], that K_6 and $K_{3,3,1}$ are intrinsically linked in S^3 , show that $\omega(\tilde{H}) = 1$.

Thus, $\omega(\tilde{G}) = 1$, and there must be disjoint circuits J and K in \tilde{G} that do not bound disjoint disks in \tilde{M} , contradicting that \tilde{f} is an unlinked embedding. Thus, $f(G)$ is linked in M . □

Let G be a graph which contains a triangle Δ . Remove the three edges of Δ from G . Add three new edges, connecting the three vertices of Δ to a new vertex. The resulting graph, G' , is said to have been obtained from G by a “ $\Delta - Y$ move” (Figure 2). The seven graphs that can be obtained from K_6 and $K_{3,3,1}$ by $\Delta - Y$ moves are the *Petersen family* of graphs (Figure 1).

If a graph G' can be obtained from a graph G by repeatedly deleting edges and isolated vertices of G , and/or contracting edges of G , then G' is a *minor* of G .

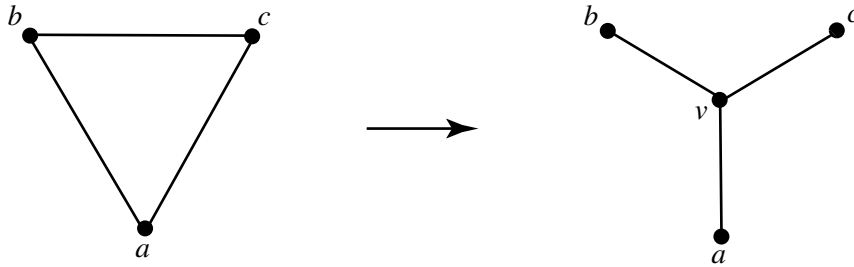


Figure 2: A $\Delta - Y$ Move

The following facts were first proved, in the S^3 case, by Motwani, Raghunathan and Saran [6]. Here we generalize the proofs to any 3-manifold M .

Fact 1 *If a graph G is intrinsically linked in M , and G' is obtained from G by a $\Delta - Y$ move, then G' is intrinsically linked in M .*

Proof Suppose to the contrary that G' has an unlinked embedding $f: G' \rightarrow M$. Let a, b, c and v be the embedded vertices of the Y illustrated in Figure 2. Let B denote a regular neighborhood of the embedded Y such that a, b and c are on the boundary of B , v is in the interior of B , and B is otherwise disjoint from $f(G')$. Now add edges ab, bc and ac in the boundary of B so that the resulting embedding of the K_4 with vertices a, b, c , and v is *panalled* in B (ie, every cycle bounds a disk in the complement of the graph). We now remove vertex v (and its incident edges) to get an embedding h of G such that if e is any edge of $G \cap G'$ then $h(e) = f(e)$ and the triangle abc is in ∂B .

Observe that if K is any circuit in $h(G)$ other than the triangle abc , then K is isotopic to a circuit in G' . The triangle abc bounds a disk in B , and since $f(G')$ is unknotted, every circuit in $f(G')$ bounds a disk in M . Thus $h(G)$ is unknotted. Also if J and K are disjoint circuits in $h(G)$ neither of which is abc , then $J \cup K$ is isotopic to a pair of disjoint circuits $J' \cup K'$ in $f(G')$. Since $f(G')$ is unlinked, J' and K' bound disjoint disks in M . Hence J and K also bound disjoint disks in M . Finally if K is a circuit in $h(G)$ which is disjoint from abc , then K is contained in $f(G')$. Since $f(G')$ is unknotted,

K bounds a disk D in M . Furthermore, since B is a ball, we can isotope D to a disk which is disjoint from B . Now abc and K bound disjoint disks in M . So $h(G)$ is unlinked, contradicting the hypothesis that G is intrinsically linked in M . We conclude that G' is also intrinsically linked in M . \square

Fact 2 *If a graph G has an unlinked embedding in M , then so does every minor of G .*

Proof The proof is identical to the proof for S^3 . \square

Theorem 1 *Let G be a graph, and let M be a 3–manifold. The following are equivalent:*

- (1) G is intrinsically linked in M ,
- (2) G is intrinsically linked in S^3 ,
- (3) G has a minor in the Petersen family of graphs.

Proof Robertson, Seymour and Thomas [10] proved that (2) and (3) are equivalent. We see as follows that (1) implies (2): Suppose there is an unlinked embedding of G in S^3 . Then the embedded graph and its system of disks in S^3 are contained in a ball, which embeds in M .

We will complete the proof by checking that (3) implies (1). K_6 and $K_{3,3,1}$ are intrinsically linked in M by Proposition 1. Thus, by Fact 1, all the graphs in the Petersen family are intrinsically linked in M . Therefore, if G has a minor in the Petersen family, then it is intrinsically linked in M , by Fact 2. \square

3 Compact subsets of a simply connected space

In this section, we assume the Poincaré Conjecture, and present some known results about 3–manifolds, which will be used in Section 4 to prove that intrinsic knotting is independent of the 3–manifold (Theorem 2).

Fact 3 *Assume that the Poincaré Conjecture is true. Let \tilde{M} be a simply connected 3–manifold, and suppose that $B \subseteq \tilde{M}$ is a compact 3–manifold whose boundary is a disjoint union of spheres. Then B is a ball with holes (possibly zero holes).*

Proof By the Seifert–Van Kampen theorem, B itself is simply connected. Cap off each boundary component of B with a ball, and the result is a closed simply connected 3–manifold. By the Poincaré Conjecture, this must be the 3–sphere. \square

Fact 4 Let \tilde{M} be a simply connected 3-manifold, and suppose that $N \subseteq \tilde{M}$ is a compact 3-manifold whose boundary is nonempty and not a union of spheres. Then there is a compression disk D in \tilde{M} for a component of ∂N such that $D \cap \partial N = \partial D$.

Proof Since ∂N is nonempty, and not a union of spheres, there is a boundary component F with positive genus. Because \tilde{M} is simply connected, F is not incompressible in \tilde{M} . Thus, F has a compression disk.

Among all compression disks for boundary components of N (intersecting ∂N transversely), let D be one such that $D \cap \partial N$ consists of the fewest circles. Suppose, for the sake of contradiction, that there is a circle of intersection in the interior of D . Let c be a circle of intersection which is innermost in D , bounding a disk D' in D . Either c is nontrivial in $\pi_1(\partial N)$, in which case D' is itself a compression disk; or c is trivial, bounding a disk on ∂N , which can be used to remove the circle c of intersection from $D \cap \partial N$. In either case, there is a compression disk for ∂N which has fewer intersections with ∂N than D has, contradicting minimality. Thus, $D \cap \partial N = \partial D$. \square

We are now ready to prove the main result of this section. Because its proof uses [Fact 3](#), it relies on the Poincaré Conjecture.

Proposition 2 Assume that the Poincaré Conjecture is true. Then every compact subset K of a simply connected 3-manifold \tilde{M} is homeomorphic to a subset of S^3 .

Proof We may assume without loss of generality that K is connected. Let $N \subseteq \tilde{M}$ be a closed regular neighborhood of K in \tilde{M} . Then N is a compact connected 3-manifold with boundary. It suffices to show that N embeds in S^3 .

Let $g(S)$ denote the genus of a connected closed orientable surface S . Define the complexity $c(S)$ of a closed orientable surface S to be the sum of the squares of the genera of the components S_i of S , so $c(S) = \sum_{S_i} g(S_i)^2$. Our proof will proceed by induction on $c(\partial N)$. We make two observations about the complexity function.

- (1) $c(S) = 0$ if and only if S is a union of spheres.
- (2) If S' is obtained from S by surgery along a non-trivial simple closed curve γ , then $c(S') < c(S)$.

We prove Observation (2) as follows. It is enough to consider the component S_0 of S containing γ . If γ separates S_0 , then $S_0 = S_1 \# S_2$, where S_1 and S_2 are not spheres, and S' is the result of replacing S_0 by $S_1 \cup S_2$ in S . In this case, $c(S_0) = g(S_0)^2 = (g(S_1) + g(S_2))^2 = c(S_1) + c(S_2) + 2g(S_1)g(S_2) > c(S_1) + c(S_2)$, since

$g(S_1)$ and $g(S_2)$ are nonzero. On the other hand, if γ does not separate S_0 , then surgery along γ reduces the genus of the surface. Then the square of the genus is also smaller, and hence again $c(S') < c(S)$.

If $c(\partial N) = 0$, then by [Fact 3](#) N is a ball with holes, and so embeds in S^3 , establishing our base case. If $c(\partial N) > 0$, then by [Fact 4](#) there is a compression disk D for ∂N such that $D \cap \partial N = \partial D$. There are three cases to consider.

Case 1 $D \cap N = \partial D$. Let $N' = N \cup \text{ncbd}(D)$. Since $\partial N'$ is the result of surgery on ∂N along a non-trivial simple closed curve, $c(\partial N') < c(\partial N)$, so by induction N' embeds in S^3 . Hence N embeds in S^3 .

Case 2 $D \cap N = D$, and D separates N . Then cutting N along D (ie removing $D \times (-1, 1)$) yields two connected manifolds N_1 and N_2 , with $c(\partial N_1) < c(\partial N)$ and $c(\partial N_2) < c(\partial N)$. So N_1 and N_2 each embed in S^3 . Consider two copies of S^3 , one containing N_1 and the other containing N_2 .

Let C_1 be the component of $S^3 - N_1$ whose boundary contains $D \times \{1\}$, and C_2 be the component of $S^3 - N_2$ whose boundary contains $D \times \{-1\}$. Remove small balls B_1 and B_2 from C_1 and C_2 , respectively. Then glue together the balls $\text{cl}(S^3 - B_1)$ and $\text{cl}(S^3 - B_2)$ along their boundaries. The result is a 3-sphere containing both N_1 and N_2 , in which $D \times \{1\}$ and $D \times \{-1\}$ lie in the boundary of the same component of $S^3 - (N_1 \cup N_2)$. So we can embed the arc $\{0\} \times (-1, 1)$ (the core of $D \times (-1, 1)$) in $S^3 - (N_1 \cup N_2)$, which means we can extend the embedding of $N_1 \cup N_2$ to an embedding of N .

Case 3 $D \cap N = D$, but D does not separate N . Then cutting N along D yields a new connected manifold N' with $c(\partial N') < c(\partial N)$, so N' embeds in S^3 . As in the last case, we also need to embed the core γ of D . Suppose for the sake of contradiction that γ has endpoints on two different boundary components F_1 and F_2 of N' . Let β be a properly embedded arc in N' connecting F_1 and F_2 . Then $\gamma \cup \beta$ is a loop in \tilde{M} that intersects the closed surface F_1 in exactly one point. But because $H_1(\tilde{M}) = 0$, the algebraic intersection number of $\gamma \cup \beta$ with F_1 is zero. This is impossible since $\gamma \cup \beta$ meets F_1 in a single point. Thus, both endpoints of γ lie on the same boundary component of N' , and so γ can be embedded in $S^3 - N'$. So the embedding of N' can be extended to an embedding of N in S^3 . \square

4 Intrinsically knotted graphs

In this section, we use [Proposition 2](#) to prove that the property of a graph being intrinsically knotted is independent of the 3-manifold it is embedded in. Notice that

since [Proposition 2](#) relies on the Poincaré Conjecture, so does the intrinsic knotting result.

Theorem 2 *Assume that the Poincaré Conjecture is true. Let M be a 3–manifold. A graph is intrinsically knotted in M if and only if it is intrinsically knotted in S^3 .*

Proof Suppose that a graph G is not intrinsically knotted in S^3 . Then it embeds in S^3 in such a way that every circuit bounds a disk embedded in S^3 . The union of the embedding of G with these disks is compact, hence is contained in a ball B in S^3 . Any embedding of B in M yields an unknotted embedding of G in M .

Conversely, suppose there is an unknotted embedding $f: G \rightarrow M$. Let \tilde{M} be the universal cover of M . By using the same argument as in the proof of Lemma 1, we can lift f to an unknotted embedding $\tilde{f}: G \rightarrow \tilde{M}$. Let K be the union of $\tilde{f}(G)$ with the disks bounded by its circuits. Then K is compact, so by [Proposition 2](#), there is an embedding $g: K \rightarrow S^3$. Now $g \circ \tilde{f}(G)$ is an embedding of G in S^3 , in which every circuit bounds a disk. Hence $g \circ \tilde{f}(G)$ is an unknotted embedding of G in S^3 . \square

Remark The proof of [Theorem 2](#) can also be used, almost verbatim, to show that intrinsic *linking* is independent of the 3–manifold. Of course, this argument relies on the Poincaré Conjecture; so the proof given in [Section 2](#) is more elementary.

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