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Limits of incompressible surfaces

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Abstract

One can embed arbitrarily many disjoint, non-parallel, non-boundary parallel, incompressible surfaces in any three manifold with at least one boundary component of genus two or greater (Howards, 1998). This paper proves the contrasting, but not contradictory result that although one can sometimes embed arbitrarily many surfaces in a 3-manifold it is impossible to ever embed an infinite number of such surfaces in any compact, orientable 3-manifold M. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction and definitions

We begin by reviewing a few definitions which can be found in most introductory texts on 3-manifolds. We rely heavily on Hempel's versions in [1].

A surface $(F, \partial F)$ that is embedded in a 3-manifold $(M, \partial M)$ is *properly embedded* if $F \cap \partial M = \partial F$. From this point on when we refer to a surface in a three-manifold, we will be talking about properly embedded surfaces unless otherwise noted. Two surfaces F_1 and F_2 in a 3-manifold M are *parallel* if they co-bound a product ($F \times I$; $F \times 0 = F_1$, $F \times 1 = F_2$) in M and $\partial F \times I \subset \partial M$. A surface F_1 in a 3-manifold M is *boundary parallel* if it co-bounds a product with F_2, a subsurface of the boundary ($F \times I$; $F \times 0 = F_1$, $F \times 1 = F_2$) and $\partial F \times I \subset \partial M$ in M. A surface F embedded in a three-manifold M is called *compressible* if any of the following apply.

(1) F is a 2-sphere which bounds a ball in M,

(2) F is a disk and either $F \subset \partial M$ or there is a ball $B \subset M$ such that $\partial B = F \cup D$ where D is a disk contained in ∂M, or

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(3) there is a disk $D \subset M$ with $D \cap F = \partial D$ and ∂D not contractible in F. (Note that D, of course, is not required to be properly embedded in M.)

Otherwise, F is *incompressible*.

A surface F is *boundary compressible* in a three-manifold M if either

(1) F is parallel to a disk in the boundary of M or

(2) there exists a disk D in M such that $D \cap F = \alpha$, an arc in ∂D, and $D \cap \partial M = \beta$ is an arc in ∂D with $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D$, and either α (β) does not separate F $(\partial M - \partial F)$ or α separates F $(\partial M - \partial F)$ into two components and the closure of neither is a disk.

Otherwise, F is *boundary incompressible*.

A 3-manifold, M is *irreducible* if every embedded 2-sphere in M bounds a 3-ball.

We end with an algebraic definition. Let $G = G_1 *_H G_2$ designate the *free product with amalgamation* of the groups G_1 and G_2 over the group H.

2. The free product with amalgamation

Howards [4] demonstrates that the free group on two generators may be split into a free product with amalgamation over two arbitrarily large free groups. This section proves the following contrasting result (it is probably known but does not seem to have ever been written down):

Theorem 2.1. *Let G_1 be a group that is not finitely generated and let H be finitely generated, then $G = G_1 *_H G_2$ is not finitely generated.*

Proof. Let M_i be a $K(\pi, 1)$ for G_i $(i = 0$ or $1)$. Let S_i be the image of S a complex that maps in to represent H in each M_i. Connect S_0 to S_1 with a cylinder $A = S \times I$ with $S \times 0 = S_0$ and $S \times 1 = S_1$ yielding a new space, M. This gives us the free product with amalgamation for which we are searching.

Choose a base point on $S' = S \times 1/2$ for $\pi_1(M)$. If $\pi_1(M)$ is finitely generated, then choose a set of generators, $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i, \ldots, \alpha_n\}$, where $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$ generate H. Choose generators for $\pi_1(M_1)$ $\{\gamma_1, \ldots, \gamma_{i-1}, \gamma_i, \ldots\}$ and generators $\pi_1(M_2)$, $\{\beta_1, \ldots, \beta_{i-1}, \beta_i, \ldots\}$ where $\{\gamma_1, \ldots, \gamma_{i-1}\}$ (and $\{\beta_1, \ldots, \beta_{i-1}\}$) are $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$ pushed off into M_1 (and M_2, respectively).

Now, $\{\alpha_i, \ldots, \alpha_n\}$ may weave back and forth between M_1 and M_2 a finite number of times and may be expressed as

$$\{(\gamma_{i_1}^{\pm}\beta_{i_1}^{\pm} \ldots \gamma_{i_j}^{\pm}\beta_{i_j}^{\pm}), \ldots, (\gamma_{n_1}^{\pm}\beta_{n_1}^{\pm} \ldots \gamma_{n_k}^{\pm}\beta_{n_k}^{\pm})\},$$

where each γ_{s_m}, $i \leqslant s \leqslant n$, (or β_{s_m}) is either some γ_i or e (or some β_i or e). Examine any γ_l in $\pi_1(M_1)$. It must be in the span of $\{\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i_1}, \ldots, \gamma_{n_k}\}$.

To see this, take a disk D bounded by γ_l followed by γ_l^{-1} expressed as a product of the generators of $\pi_1(M)$ that intersects S' transversally and minimally. Since the fundamental group of S' injects, we can assume $D \cap S'$ has no simple closed curves. Examine the portion of $D - D \cap S'$ that contains γ_l. This must be a disk whose boundary consists of generators of $\pi_1(M_1)$ and curves on S' that hence may be expressed as products of $\{\gamma_1, \ldots, \gamma_{i-1}\}$ and their inverses. Therefore, γ_l may be written in terms of $\{\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i_1}, \ldots, \gamma_{n_k}\}$ and their inverses. Thus, $\pi_1(M_1)$ must be finitely generated. □

Corollary 2.2. *Given a compact orientable* 3*-manifold* M *and a set of surfaces* $\{F_i\}$ *in* M*, each of the regions in* $M - \bigcup\{F_i\}$ *has finitely generated fundamental group.*

Proof. After splitting M along a finite number of the surfaces we attain M' a compact 3-manifold with finitely generated fundamental group for which each of the remaining $\{F_i\}$ is separating. None of the complementary regions could have infinitely generated fundamental group or else M' would have to, also. □

Lemma 2.3. *The boundary of a* 3*-manifold* M *with finitely generated* $\pi_1(M)$ *must have bounded genus.*

Proof. Take a Scott core C for M and expand C to remain a compact submanifold of M, but to include an arbitrarily large portion of the boundary (for example, one might take the C' equal to C plus the closure of a neighborhood of $B \cup A$ where B is a (topologically) large portion of the boundary and A consists of arcs running from B to C). C' can be further expanded to become a Scott core C'' by adding 2-handles that kill any added elements of π_1. Now since C'' has the same fundamental group as M and H_1 is just abelianized π_1, $H_1(C'')$ must have no more generators than $\pi_1(M)$, but the boundary of C'' has arbitrarily high genus ($\partial(M) \cap C'' = \partial(M) \cap C'$) and since C'' is compact, "half lives–half dies" assures us that $H_1(C'')$ has arbitrarily many generators, which is a contradiction. □

Note. This proof actually shows that any compact submanifold of M is contained in a Scott core. We should also point out that one can also use a more complicated homology argument to prove the lemma.

3. The behavior of incompressible surfaces in a 3-manifold

For a while it was claimed that one could never embed arbitrarily many disjoint, non-parallel, non-boundary parallel, incompressible surfaces in a three-manifold. The first counter example was found in [5]. More recently a more general argument has been used to show that any manifold with at least one boundary component of at least genus two allows such embeddings [4]. On the other hand, Benedict and Mike Freedman showed that in any manifold, if the Euler Characteristic of the surfaces is bounded, then the number of surfaces will also be bounded [3]. The result in [4] contrasts with, but does not contradict the main theorem of this paper.

Theorem 3.1. *No compact, orientable three manifold can support an infinite number of disjoint, non-parallel, non-boundary-parallel, incompressible surfaces.*

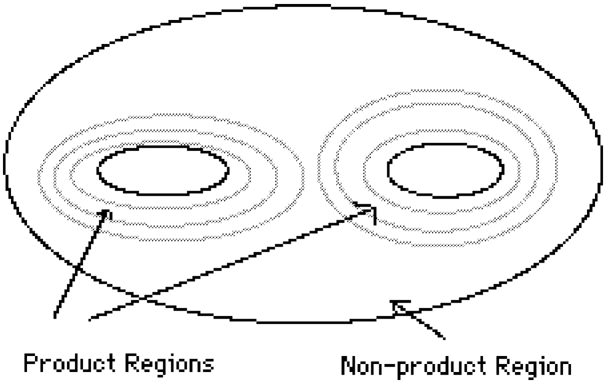

Fig. 1. Product and non-product regions.

Note. An easy argument in [3] shows that any given three manifold supports only a bounded number of boundary-parallel, but non-parallel surfaces, if it is assumed that none of the surfaces are disks or annuli. Thus, this assumption could replace the non-boundary parallel assumption above.

Proof. To begin our proof, we should recall that the usual Haken finiteness argument using normal surfaces shows that there can only be a finite number of incompressible, boundary-incompressible surfaces, so we need only consider the surfaces that are incompressible but boundary compressible.

Given this, in order to derive a contradiction we may assume that we have an infinite list of surfaces $\{F_i\}$. It will be convenient later to assume we have no annuli, and [3] assures us that M can only have a finite number of possible disjoint non-boundary-parallel annuli, therefore that we may assume that none of the surfaces on the infinite list of $\{F_i\}$ are annuli. Note that M only has a finite number of boundary components. Also note that since each F_i is boundary compressible, each one meets at least one boundary component of M in a set of simple closed curves. We will choose one of the surfaces and examine its boundary compression disk.

We examine how the $\{F_i\}$ intersect the boundary of M and define a product region in ∂M to be an annular region with boundary two parallel curves from the boundary of the $\{F_i\}$. Of course a non-product region simply refers to a region of the boundary which is not a product region. See Fig. 1. □

Lemma 3.2. *The boundary components of the surfaces F_i are each parallel to one of a finite number of curves on the boundary of M.*

Proof. Since M has a finite number of boundary components and all of the F_i are disjoint and therefore have disjoint boundary components, the proof is an easy application of hierarchy arguments to nontrivial simple closed curves on closed orientable surfaces. □

Since there are only a finite number of curve types on the boundary there can only be a finite number of non-product regions. These in kind can only correspond to $\{M_1, \ldots, M_n\}$, a finite subset of the regions obtained by cutting M up along the union of the $\{F_i\}$.

Let M_i' be a closed regular neighborhood in M_i of the boundary of M_i. This means that M_i' is a (not necessarily connected) surface crossed with the unit interval and therefore has incompressible boundary.

Now in M replace M_i by M_i' obtaining M'. Since we have just observed that the $\{M_i\}$ is a finite set of pieces and the pieces that do not have non-product regions on their boundary are left alone and not replaced, we have only altered a finite number of regions. Corollary 2.2 together with Lemma 2.3 assures us that for any region M_i, ∂M_i has bounded genus. We also note that if the boundary of a 3-manifold has a "puncture", the puncture has to extend to infinity, so a neighborhood of the puncture must be an infinitely long annulus. Since the boundary of M_i is made from a (potentially infinite) list of compact surfaces, such a puncture could only result from an infinite list of annuli glued together, but this is impossible since none of the $\{F_i\}$ are annuli. M was compact, and each of the new pieces are compact, therefore M' is, too.

The infinite collection of disjoint, non-parallel, non-boundary-parallel, incompressible surfaces in M become an infinite collection of disjoint, non-parallel, non-boundary-parallel, incompressible surfaces in M'. The remainder of the paper will show that the surfaces are also boundary-incompressible in M' which is a contradiction.

We now choose an F_i and examine a disk representing its boundary compression. Choose the outermost such boundary compression disk with respect to F_i, so that the interior of the disk is disjoint from F_i. We look at the intersection of the disk with the other surfaces. Since the surfaces are incompressible, we may use an innermost loop argument to show that we can choose to have only arcs and no simple closed curves in the intersection.

Lemma 3.3. *The boundary of boundary compressing disks runs through a non-product region.*

Proof. If the component were strictly in product regions, either the component would connect a boundary component to itself in a trivial manner which is prohibited by the definition of a boundary compression, or it would cross the annulus in the unique arc connecting the two boundary components. (See Fig. 2.)

In the latter case, after compressing we have a boundary component which is trivial in the fundamental group of ∂M. So now we have two options: either we have a disk, which is impossible since that would mean we started with an annulus, or else we have a compressible surface. This is also a contradiction as performing a boundary compression cannot make a surface compressible that was not already compressible.

The boundary compressing disk must therefore intersect a non-product region essentially. This yields a compressing disk for a boundary component of one of the M_i' (since those are the only pieces which can have a non-product region on its boundary), but these pieces have incompressible boundary, so this is a contradiction. There is no boundary compressing disk and therefore the surfaces must be boundary incompressible in M'.

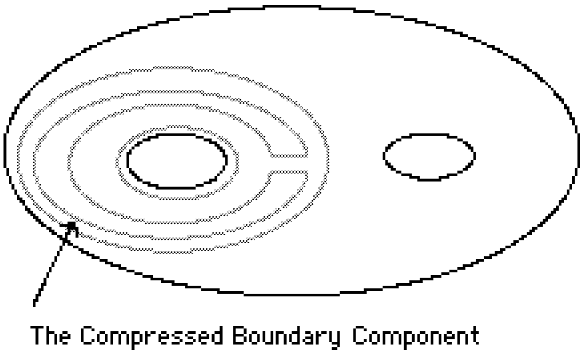

Fig. 2. A boundary compression in a product region.

Haken finiteness now applies, so there are only a finite number of surfaces. This contradicts our original assumption. □

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