Geometry & Topology Monographs 12 (2007) 89–120
arXiv version: fonts, pagination and layout may vary from GTM published version

Thin position for knots and 3–manifolds: a unified approach

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We unify the notions of thin position for knots and for 3–manifolds and survey recent work concerning these notions.

57M27, 57M25; 57N10

1 Introduction

Thin position for knots and for 3–manifolds have become basic tools for 3–manifold topologists and knot theorists. When David Gabai first introduced the notion of thin position for knots as an ad hoc tool in studying foliations of 3–manifolds he may not have foreseen the widespread interest this notion would engender. Thin position for knots featured prominently in the work of Mark Culler, Cameron McA Gordon, John Luecke and Peter Shalen concerning Dehn surgery on knots as well in the proof by Cameron McA Gordon and John Luecke that knots are determined by their complements. It also played a crucial role in Abigail Thompson's proof that there is an algorithm to recognize $\mathbb{S}^3$; Rubinstein's original argument [20] used the related concept of *minimax sweepouts* and *normal surfaces*.

A knot in thin position appears to be ideally situated from many points of view. This is demonstrated, for instance, by the work of Daniel J. Heath and Tsuyoshi Kobayashi. There is also a growing expectation that some knot invariants can be calculated most efficiently by employing thin position.

Later, Martin Scharlemann and Abigail Thompson introduced a related, but not completely analogous, notion of thin position for 3–manifolds. At first glance, their theory appeared elegant but of little use. It took a number of years for the strength of their theory to come to fruition. This theory has now become one of the fundamental tools in the study of 3–manifolds. Moreover, it has proved more natural than the notion of thin position for knots. This has prompted Martin Scharlemann and Abigail Thompson to begin reworking the notion of thin position for knots under the guise of "slender knots". Their work is beyond the scope of this article.

Published: 3 December 2007 DOI: 10.2140/gtm.2007.12.89

The aim of this article is to introduce the novice to the notion of thin position for knots and 3–manifolds. The emphasis here is to underline the formal analogy of the definitions. Each of these notions is defined more naturally elsewhere. For the most natural definition of thin position for knots, see Gabai [2]. And for a more extensive treatment of thin position for knots, see Scharlemann [22]. For the most natural definition of thin position for 3–manifolds, see Scharlemann and Thompson [26]. The added formality here is designed to unify the two definitions. This should allow an easy adaptation of the underlying framework to numerous other settings. In this paper we avoid some of the more technical details; for an extensive introduction to the subject see Saito, Scharlemann and Schultens [21].

We wish to thank Dave Bayer for suggesting this project, Marty Scharlemann for helpful discussions, and the referee for many helpful suggestions.

2 Thin position

To define thin position in a general setting, we need the following: A pair of manifolds (N, M) with $N \subset M$. A constraint C that may be placed on the set, $\mathcal{M}$, of Morse functions on (N, M). A function $g\colon \mathcal{L} \to \mathbb{R}^\infty$, for $\mathcal{L}$ the set of ordered pairs of level sets of the elements of $\mathcal{M}$. A well ordered set $\mathcal{O}$. And finally, a function $f\colon \mathbb{R}^\infty \to \mathcal{O}$. We note that g maps into $\mathbb{R}^n$ (for some n that depends on the manifold and the knot); we identify $\mathbb{R}^n$ with $\mathbb{R}^\infty$ with all but the first n coordinates set to zero. Intuitively, g measures the complexity of individual levels and f measures the complexity of (N, M).

Remark 2.1 In fact, the definition can be made a little more general, as N does not need to be a manifold. As an example, below we discuss a few settings were N is a graph.

Let (N, M), C, g, $\mathcal{O}$ and f be as required. Set

$$\mathcal{C} = \{h \in \mathcal{M} \mid h \text{ satisfies } C\}.$$

Given $h \in \mathcal{C}$, denote the critical values of h, in increasing order, by $c_0, \dots, c_n$. Note that since h is a Morse function on pairs, a critical value of h is a critical value either of $h|_N$ or of $h|_M$. For $i = 1, \dots, n$, choose a regular value r_i such that $c_{i-1} < r_i < c_i$. Consider the finite sequence

$$(h|_N^{-1}(r_1), h|_M^{-1}(r_1)) \dots, (h|_N^{-1}(r_n), h|_M^{-1}(r_n))$$

of ordered pairs of level sets of h and the corresponding ordered 2n-tuple

$$(g(h|_N^{-1}(r_1), h|_M^{-1}(r_1)), \ldots, g(h|_N^{-1}(r_n), h|_M^{-1}(r_n))) \in \mathbb{R}^n \in \mathbb{R}^\infty.$$

Set

$$w_h(N) = f(g(h|_N^{-1}(r_1), h|_M^{-1}(r_1)), \ldots, g(h|_N^{-1}(r_n), h|_M^{-1}(r_n))).$$

We call $w_h(N)$ the width of N relative to h. Set

$$w(N) = \min\{\ w_h(N) \mid h\ \in\ \mathcal{C}\}.$$

We call $w(N)$ the width of $(N, C, g, \mathcal{O}, f)$. We say that $(N, C, g, \mathcal{O}, f)$ is in thin position if it is presented together with $h \in \mathcal{C}$ such that $w(N) = w_h(N)$.

If r_i is such that

$$g(h|_N^{-1}(r_{i-1}), h|_M^{-1}(r_{i-1})) < g(h|_N^{-1}(r_i), h|_M^{-1}(r_i)) > g(h|_N^{-1}(r_{i+1}), h|_M^{-1}(r_{i+1}))$$

where $<$ and $>$ are in the dictionary order, then we call $(h|_N^{-1}(r_i), h|_M^{-1}(r_i))$ a thick level. If r_i is such that

$$g(h|_N^{-1}(r_{i-1}), h|_M^{-1}(r_{i-1})) > g(h|_N^{-1}(r_i), h|_M^{-1}(r_i)) < g(h|_N^{-1}(r_{i+1}), h|_M^{-1}(r_{i+1}))$$

in the dictionary order, then we call $(h|_N^{-1}(r_i), h|_M^{-1}(r_i))$ a thin level.

2.1 Thin position for knots

The notion of thin position for knots was introduced by D. Gabai. He designed and used this notion successfully to prove Property R for knots. We here specify (N, M), C, g, $\mathcal{O}$ and f as used in the context of thin position for knots. Let

$$(N, M) = (K, \mathbb{S}^3)$$

be a knot type. Take C to be the requirement that the Morse function $h\colon (K, \mathbb{S}^3) \to \mathbb{R}$ has exactly two critical points on $\mathbb{S}^3$ (a maximum, ∞, and a minimum, $-\infty$); we call such a function a (standard) height function of $\mathbb{S}^3$. In considering thin position for knots, we may visualize our Morse function as projection onto the vertical coordinate. The fact that we may do so derives from the constraint placed on the Morse functions under consideration.

Let g be the function that takes the ordered pair

$$(h|_K^{-1}(r_i), h^{-1}(r_i))$$

of level sets of a Morse function h to

$$\chi(h|_K^{-1}(r_i))$$

And let $\mathcal{O}$ be $\mathbb{N}$ and $f\colon \mathbb{R}^\infty \to \mathbb{N}$ the function defined by

$$f(x_1,\dots,x_n) = \sum_i x_i$$

Thus in this case, we proceed as follows: Given a Morse function $h\colon (K,\mathbb{S}^3) \to \mathbb{R}$ of pairs such that $h|_M$ is a height function, let $c_0,\dots,c_n$ be the critical points of h. Note that since these critical points are critical points of either $h|_K$ or of $h|_{\mathbb{S}^3}$, exactly two of these critical points will be critical points of $h|_{\mathbb{S}^3}$. Note further that one of these critical points lies below all critical points of $h|_K$ and the other lies above all critical points of $h|_K$.

Now, for $i = 1,\dots,n$, choose regular values r_i such that $c_{i-1} < r_i < c_i$. Consider pairs of level surfaces

$$(h|_K^{-1}(r_i), h|_{\mathbb{S}^3}^{-1}(r_i))$$

and

$$g(h|_K^{-1}(r_i), h|_{\mathbb{S}^3}^{-1}(r_i)) = \chi(h|_K^{-1}(r_i)) = \#|K \cap (h|_{\mathbb{S}^3})^{-1}(r_i)|$$

Note that here

$$h|_K^{-1}(r_1)) = h|_K^{-1}(r_n)) = \emptyset$$

and thus

$$g(h|_K^{-1}(r_1), h|_{\mathbb{S}^3}^{-1}(r_1)) = g(h|_K^{-1}(r_n), h|_{\mathbb{S}^3}^{-1}(r_n)) = 0.$$

This yields the ordered n-tuple

$$(0, \#|K \cap (h|_{\mathbb{S}^3})^{-1}(r_2)|, \dots, \#|K \cap (h|_{\mathbb{S}^3})^{-1}(r_{n-1})|, 0)$$

And thus

$$w_h(K,\mathbb{S}^3) = 0 + \#|K \cap (h|_{\mathbb{S}^3})^{-1}(r_2)| + \cdots + \#|K \cap (h|_{\mathbb{S}^3})^{-1}(r_{n-1})| + 0$$

In Figure 1, the knot pictured schematically has

$$w_h(K,\mathbb{S}^3) = 0+2+4+6+4+6+8+6+4+2+0 = 42$$

The width of $(K,\mathbb{S}^3)$ is the smallest possible relative width $w_h(K)$, as h ranges over all height functions on $\mathbb{S}^3$. In the usual computation of width, one considers only critical points of $h|_K$, one thus considers two fewer critical points and two fewer regular points and is thus not compelled to add the 0's in the sum.

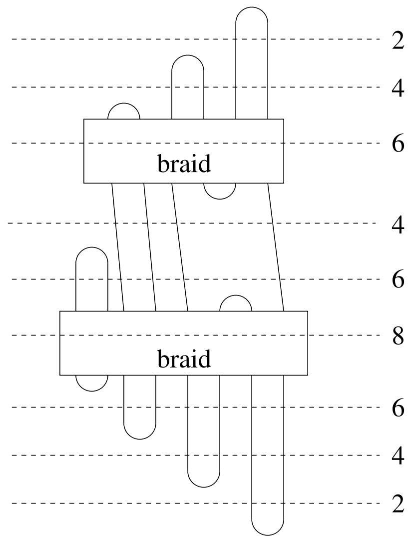

Figure 1: Thin position for knots

2.2 Thin position for 3–manifolds

The notion of thin position for 3–manifolds was pioneered by Scharlemann and Thompson. We here specify (N, M), C, g, $\mathcal{O}$ and f as used in the context of thin position for 3–manifolds. Let $N = M$ and let M be a closed 3–manifold. Let C be the vacuous requirement (we consider all Morse functions). Let g be the function that takes the ordered pair

$$(h^{-1}(r_i), \emptyset)$$

of level sets of a Morse function h to

$$\#|h^{-1}(r_i)| + s_i - \chi(h^{-1}(r_i)),$$

where s_i is the number of S^2 components in $h^{-1}(r_i)$. Let $\mathcal{O}$ be $\mathbb{N}^\infty$ in the dictionary order. Finally, let $f\colon \mathbb{R}^\infty \to \mathbb{Z}^\infty$ be the function that takes the ordered n–tuple $(x_1, \ldots, x_n)$, deletes all entries x_i for which either $x_{i-1} > x_i$ or $x_{i+1} > x_i$ and then arranges the remaining entries (that is, the local maxima) in nonincreasing order.

Thus in this case, we proceed as follows: We identify (M, M) with M. Let h be a Morse function

$$h\colon M \to \mathbb{R}.$$

Let $c_0, \ldots, c_n$ be the critical points of h and for $i = 1, \ldots, n$, choose regular values r_i

such that $c_{i-1} < r_i < c_i$. Consider the level surfaces

$$h^{-1}(r_1), \cdots, h^{-1}(r_n)$$

and

$$g(h^{-1}(r_i)) = \#|h^{-1}(r_i)| + s_i - \chi(h^{-1}(r_i))$$

where s_i is the number of spherical components of $h^{-1}(r_i)$. This yields the ordered n-tuple

$$(\#|h^{-1}(r_1)| + s_1 - \chi(h^{-1}(r_1)), \cdots, \#|h^{-1}(r_n)| + s_n - \chi(h^{-1}(r_n))).$$

The function f picks out the values $\#|h^{-1}(r_1)| + s_1 - \chi(h^{-1}(r_1))$ for the thick levels of h and arranges them in non increasing order. Thus

$$w_h(K, \mathbb{S}^3) = f((\#|h^{-1}(r_1)| + s_1 - \chi(h^{-1}(r_1)), \cdots, \#|h^{-1}(r_n)| + s_n - \chi(h^{-1}(r_n))))$$

and $w(N)$ is the smallest such sequence arising for a Morse function h on M, in the dictionary order.

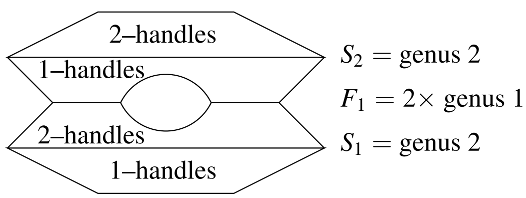

Figure 2: Thin position for 3–manifolds

The schematic in Figure 2 describes a decomposition of the 3–torus

$$\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1.$$

Note that a torus or a sphere will never appear as a thick level in a thin presentation of $\mathbb{T}^3$, and a single genus 2 surface is insufficient. So the width of $\mathbb{T}^3$ is:

$$w_h(\mathbb{T}^3) = (3, 3).$$

2.3 Thin position for knots in 3–manifolds

We here suggest a more general application of the notion of thin position to knots in 3–manifolds. This notion differs from the standard notion of thin position for knots in $\mathbb{S}^3$ in that we do not restrict our attention to specific height functions. In the setting of 3–manifolds we wish to pick Morse functions optimal with respect to both the 3–manifold and the knot.

Remark 2.2 The first application of thin position for knots in general manifolds was given in 1997 in two independent PhD dissertations: Feist (unpublished) and Rieck [16]. However, their approach is different from ours and is described below. An similar approach to the one presented here can be found in Hayashi and Shimokawa [4].

We here specify the (N, M), C, g, $\mathcal{O}$ and f we have in mind. Let M be a closed 3–manifold and let $N = K$ be a knot contained in M. Let C be the vacuous requirement. Let g be the function that takes the ordered pair

$$((h|_K)^{-1}(r_i), h^{-1}(r_i))$$

of level sets of a Morse function h to

$$2\#|h^{-1}(r_i)| - \chi(h^{-1}(r_i)) + 2\#|h|_K^{-1}(r_i)| - \chi(h|_K^{-1}(r_i)).$$

(Here the last two terms just count the number of points $\#|h|_K^{-1}(r_i)|$. The cumbersome notation aims to emphasize the equal weight of the 3–manifold and the knot.) And let $\mathcal{O}$ be $\mathbb{N}^\infty$ in the dictionary order. Finally, let $f\colon \mathbb{R}^\infty \to \mathbb{N}^\infty$ be the function that takes the ordered n–tuple $(x_1, \ldots, x_n)$ and rearranges the entries so they are in nonincreasing order.

Thus in this case, we proceed as follows: Given a Morse function

$$h\colon (K, M) \to \mathbb{R},$$

let $c_0, \ldots, c_n$ be the critical points of h. For $i = 1, \ldots, n$, choose regular values r_i such that $c_{i-1} < r_i < c_i$. Consider the pairs

$$(h|_K^{-1}(r_1), h^{-1}(r_1)), \cdots, (h|_K^{-1}(r_n), h^{-1}(r_n))$$

then

$$g((h|_K^{-1}(r_1), h^{-1}(r_i))) = 2\#|h^{-1}(r_i)| - \chi(h^{-1}(r_i)) + 2\#|h|_K^{-1}(r_i)| - \chi(h|_K^{-1}(r_i)).$$

This yields the ordered n-tuple

$$\begin{gathered}(2\#|h^{-1}(r_1)| - \chi(h^{-1}(r_1)) + 2\#|h|_K^{-1}(r_1)| - \chi(h|_K^{-1}(r_1)), \cdots, \\ 2\#|h^{-1}(r_n)| - \chi(h^{-1}(r_n)) + 2\#|h|_K^{-1}(r_n)| - \chi(h|_K^{-1}(r_n)))\end{gathered}$$

The function f rearranges the entries in non increasing order. Thus

$$\begin{aligned} w_h(K, M) &= f((2\#|h^{-1}(r_1)| - \chi(h^{-1}(r_1)) + 2\#|h|_K^{-1}(r_1)| - \chi(h|_K^{-1}(r_1)), \cdots, \\ &\qquad 2\#|h^{-1}(r_n)| - \chi(h^{-1}(r_n)) + 2\#|h|_K^{-1}(r_n)| - \chi(h|_K^{-1}(r_n)))) \end{aligned}$$

and $w(K, M)$ is the smallest such sequence arising for Morse functions on (K, M), in the dictionary order.

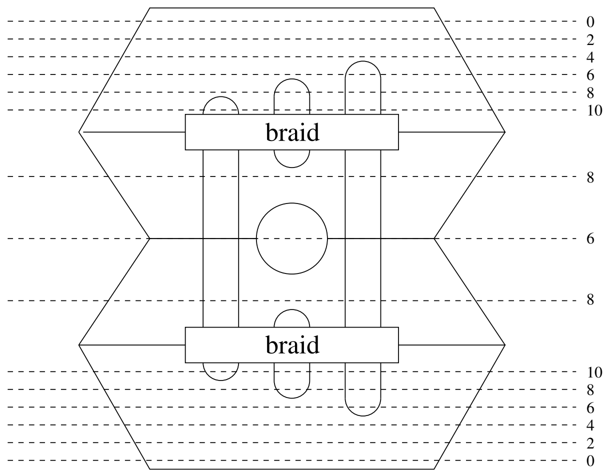

Figure 3: Thin position for knots in 3–manifolds

The schematic in Figure 3 gives $g((h^{-1}(r_i), h|_K^{-1}(r_i)))$ for a knot in $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ with respect to a specific Morse function. Here

$$w_h(K, \mathbb{T}^3) = (10, 10, 8, 8, 8, 8, 6, 6, 6, 4, 4, 2, 2, 0, 0).$$

2.4 Other settings

There are other settings to which our general theory applies. We will not work them out in detail here. One which deserves to be mentioned is that of manifolds with boundary. This setting has been studied along with the case of closed 3–manifolds as in Section 2.2. But in those studies, the functions considered are not in fact Morse functions, but rather Morse functions relative boundary, that is, functions that are Morse functions except that they are constant on boundary components.

One can consider the setting in which this requirement is dropped. Then (N, M), C, g, $\mathcal{O}$ and f are as follows: M is a 3–manifold and $N = M$ (as above we identify (M, M) with M). There are no requirements on the Morse functions (except that they be Morse functions, in particular, transverse to ∂M). And g is the function that takes the ordered pair

$$(\emptyset, h^{-1}(r_i))$$

of level sets of a Morse function h to

$$\#|h^{-1}(r_i)| + s_i - \chi(h^{-1}(r_i)),$$

where s_i is the number of spheres in $h^{-1}(r_i)$ and $\mathcal{O}$ is $\mathbb{N}^\infty$ in the dictionary order. Finally, $f\colon \mathbb{R}^\infty \to \mathbb{N}^\infty$ is the function that takes the ordered n–tuple $(x_1, \dots, x_n)$, deletes all entries x_i for which either $x_{i-1} > x_i$ or $x_{i+1} > x_i$ and then arranges the remaining entries in nonincreasing order. Much of the theory of Scharlemann and Thompson should carry over to this setting.

As mentioned above, the definition of thin position for a knot K in a 3–manifold M given by Feist and Rieck [16] is different than the definition above. It does not take into account critical points of the manifold. We can retrieve it by considering Morse functions with the following constraints: all the critical points of M of index zero or one are in $h^{-1}(-\infty,-1)$, all the critical points of M of index two or three are in $h^{-1}(1,\infty)$, and the knot in contained in $h^{-1}(-1,1)$. The width is then calculated as in $\mathbb{S}^3$ by summing the number of times K intersects each level:

$$w_h(K,M) = 0 + \#|K \cap (h|_M)^{-1}(r_2)| + \cdots + \#|K \cap (h|_M)^{-1}(r_{n-1})| + 0$$

Another important setting is graphs embedded in 3–manifolds. Although this paper is about knots and 3–manifolds, we can generalize the definition of thin position by allowing N to be a graph. A simple application of this was given by Rieck and Sedgwick [18] where the authors considered a bouquet of circles (that is, a connected graph with a single vertex). The constraint imposed is equivalent to: all the critical points of M of index zero or one are in $h^{-1}(-\infty,-1)$, all the critical points of M of index two or three are in $h^{-1}(1,\infty)$, the vertex is at level 1, and the interiors of all the edges are in $h^{-1}(-1,1)$. Again, the width was calculated as above. A more sophisticated approach was taken by Scharlemann and Thompson [25] and Goda, Scharlemann and Thompson [3], who considered trivalent graphs (that is, graphs with vertices of valence 3 only) in $\mathbb{S}^3$. They used the standard height function on $\mathbb{S}^3$. Roughly speaking, they treated a vertex as a critical point. Generically, every vertex has two edges above and one below (a Y vertex) or two edges below and one above (a λ vertex). The treatment of Y vertices is similar to that of minima and of λ vertices to that of maxima.

3 A counting argument (or why forgetfulness is practically irrelevant)

In this section we discuss a counting argument that relates two different widths if these widths are computed identically except at the final stage. That is, if (N,M), C and g are identical, but $\mathcal{O}$ and f differ in a prescribed way.

As a warm up, consider the lemma below. It is based on a comment by Clint McCrory. We say that a knot K in $\mathbb{S}^3$ is in bridge position with respect to the height function h, if all its maxima occur above all its minima. The bridge number of K is the smallest possible number of maxima as h ranges over all height functions on $\mathbb{S}^3$ (see Schultens [31]). In Section 6 we give a more detailed discussion of bridge position and its relationship to thin position.

Lemma 3.1 (Clint McCrory) *Let K be a knot in $\mathbb{S}^3$. If thin position is necessarily bridge position and the bridge number of K is n, then $w(K) = 2n^2$.*

Proof Suppose the knot is in thin position with respect to h and is also in bridge position. Then the knot has k maxima and k minima, for $k \geq n$. If we denote the critical values in increasing order by $c_0, \ldots, c_{k-1}, c_k, \ldots, c_{2k}$, then $c_0, \ldots, c_{k-1}$ are minima and $c_k, \ldots, c_{2k}$ are maxima. Thus

$$h|_K^{-1}(r_1) = 2, h|_K^{-1}(r_2) = 4, \cdots, h|_K^{-1}(r_k) = 2k$$
$$h|_K^{-1}(r_{k+1}) = 2k-2, h|_K^{-1}(r_{k+2}) = 2k-4, \cdots, h|_K^{-1}(r_{2k}) = 2.$$

See Figure 4. There each dot corresponds to $\frac{h|_K^{-1}(r_i)}{2}$ in the case where $k = 5$.

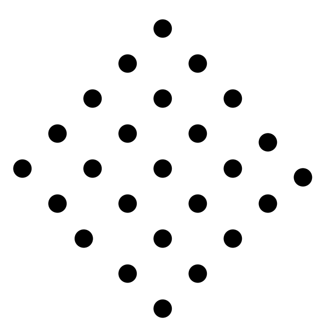

Figure 4: Calculating width

Note how the total number of dots is k^2. (This is merely a geometric visualization of the Gauss summation formula.) Thus $w_h(K) = 2k^2$. Now since bridge number is n, we see that h can be chosen so that $w_h(K) = 2n^2$. Since thin position is necessarily bridge position, it follows that $w(K) = 2n^2$. $\square$

A slightly more general version of this lemma allows us to compute the width of a knot from the thick and thin levels of a knot in thin position. This more general lemma was included in Scharlemann and Schultens [24].

Lemma 3.2 *Let $S_{i_1}, \dots, S_{i_k}$ be the thick levels of K and $F_{j_1}, \dots, F_{j_{k-1}}$ the thin levels. Set $a_{i_l} = \frac{|K \cap S_{i_l}|}{2}$ and $b_{j_l} = \frac{|K \cap F_{i_l}|}{2}$. Then*

$$w(K) = 2\sum_{l=1}^{k} a_{i_l}^2 - 2\sum_{l=1}^{k-1} b_{j_l}^2.$$

Proof We prove this by repeated use of the Gauss Summation Formula. In particular, we use the Gauss summation formula on the squares arising from thick levels. Then note that when we do so, we count the small squares arising from the thin levels twice. To compensate, we subtract the appropriate sums. See Figure 5. □

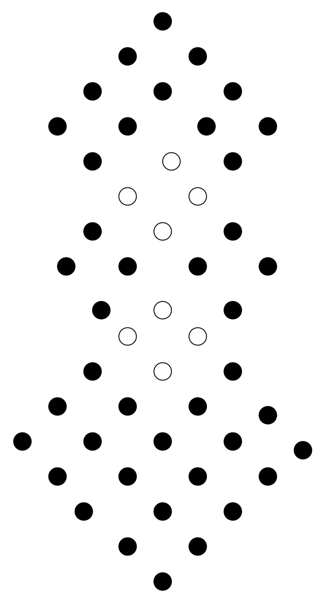

Figure 5: A cancellation principle

One consequence of this Lemma is the following: When defining thin position for knots in the traditional way as above, the relevant information is captured in the thick and thin levels. An alternate definition would thus be to use $\mathbb{N}^\infty$ instead of $\mathbb{Z}$ for $\mathcal{O}$ and to let f be the function that picks out $g(h|_K^{-1}(r_i))$ for the thick and thin levels. This would be slightly more informative than the traditional definition. Then, if f also rearranges the remaining entries in non increasing order, we lose information. In the applications of thin position of knots to the study of 3–manifolds these subtleties in the definitions appear to be irrelevant.

4 Key features of thin position

The notion of thin position was introduced by D. Gabai with a specific purpose in mind. It provided a way of describing a positioning of knots in $\mathbb{S}^3$ that made certain arguments about surfaces in the knot exterior possible. The key feature of thin position for a knot lies in the absence of disjoint pairs of upper and lower disks with respect to a regular value r of a Morse function: An upper (lower) disk for a knot K with respect to the regular level r of a Morse function h is a disk D whose boundary decomposes into two arcs, α and β, such that $\alpha \in K$, $\beta \in h^{-1}(r)$ and such that $h(a) > h(r)$ ($h(a) < h(r)$) for all a in the interior of α. We emphasize that parts of the interior of a upper (lower) disk may be below (above) $h^{-1}(r)$.

Now suppose that K is in thin position with respect to the Morse function h. Further suppose that D is an upper disk for K with respect to r and E is a lower disk for K with respect to r. If $D \cap E = \emptyset$, then we may isotope the portion of K in ∂D just below $h^{-1}(r)$ and the portion of K in ∂E just above $h^{-1}(r)$ to obtain a presentation of K that intersects $h^{-1}(r)$ four fewer times. See Figure 6 and Figure 7. It follows that after this isotopy the width is reduced by exactly four if K has exactly one maximum on ∂D above $h^{-1}(r)$ and exactly one minimum on ∂E below $h^{-1}(r)$; if K has more critical points on ∂D above $h^{-1}(r)$ or ∂E below $h^{-1}(r)$ the width is reduced by more than four. (Note that if D dips below $h^{-1}(r)$ or E above it, during the isotopy the width may increase, temporarily.)

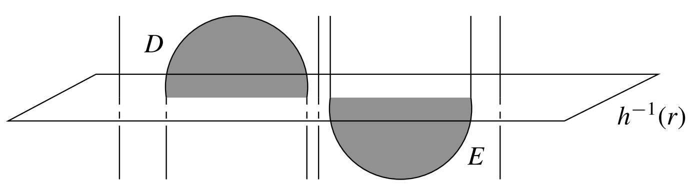

Figure 6: Two disks describing an isotopy

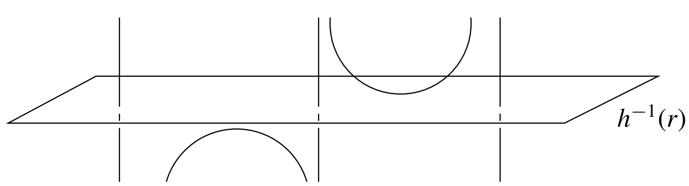

Figure 7: After the isotopy

To make sense out of this isotopy from the point of view of thin position, note that we may instead keep K fixed and alter h in accordance with the isotopy. We obtain a new

Morse function h' that coincides with h outside of a neighborhood of $D \cup E$ and such that

$$w_{h'}(K) \leq w_h(K) - 4.$$

But this contradicts the fact that K is in thin position with respect to h.

The situation is similar if $D \cap E$ consists of one point. There the relative width can be reduced by a count of 2 or more. See Figure 8 and Figure 9.

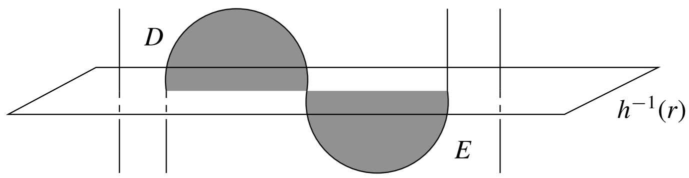

Figure 8: Two disks describing an isotopy

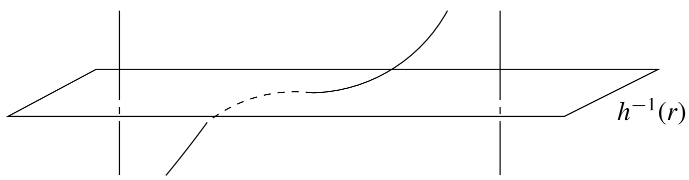

Figure 9: After the isotopy

Finally, consider the case in which $D \cap E$ consists of two points. Then one subarc of K lies in ∂D, another in ∂E and the two meet in their endpoints. It follows that K can be isotoped into the level surface $h^{-1}(r)$. In the context of knots in $\mathbb{S}^3$, $h^{-1}(r)$ is a 2–sphere and it then follows that K is trivial.

In the applications of thin position for knots to problems in 3–manifold topology the key feature used is the absence of disjoint upper and lower disks with respect to a regular value. This property is termed *locally thin* by D J Heath and T Kobayashi who investigate this property in [6].

When M Scharlemann and A Thompson introduced their notion of thin position for 3–manifolds in [26], they established a number of properties enjoyed by a 3–manifold in thin position. Let M be a 3–manifold in thin position with respect to the Morse function h. An upper (lower) compressing disk with respect to the regular value r is a disk whose boundary is an essential curve in $h^{-1}(r)$ whose interior, near ∂D lies above (below) $h^{-1}(r)$; we further impose that int$D \cap h^{-1}(r)$ consists entirely of curves that are inessential in $h^{-1}(r)$. This is analogous to an upper (lower) disk dipping below (above) the level $h^{-1}(r)$. Note that since the curves of $\text{int}D \cap h^{-1}(r)$ are inessential in $h^{-1}(r)$, an upper (lower) disk D may be isotoped (relative to the boundary) to lie entirely above (below) $h^{-1}(r)$. However, this flexibility built into this somewhat cumbersome definition is necessary for some applications.

This gives an analogy with the situation for knots in $\mathbb{S}^3$: If there are upper and lower disks with respect to r, then their boundaries must intersect.

In fact, 3–manifolds in thin position enjoy a broader spectrum of properties. Some of these can be phrased in the language of Heegaard splittings. A *compression body* W is a 3–manifold obtained from a closed (and possibly empty) surface $\partial_- W$ by taking $\partial_- W \times I$ (and, perhaps, some balls) and attaching 1–handles along $\partial_- W \times \{1\} \subset \partial_- W \times I$, where $I = [0, 1]$, and the boundaries of the balls. Then $\partial_- W$ is identified with $\partial_- W \times \{0\}$ and $\partial W \setminus \partial_- W$ is denoted $\partial_+ W$. Dually, a compression body is obtained from a connected surface $\partial_+ W$ by attaching 2–handles to $\partial_+ W \times \{0\} \subset \partial_+ W \times I$ and 3–handles to any resulting 2–spheres. A Heegaard splitting of a closed 3–manifold M is a decomposition, $M = V \cup_S W$, into two handlebodies, V, W, such that $S = \partial_+ V = \partial_+ W$. A Heegaard splitting $M = V \cup_S W$ is strongly irreducible if for any disk $(D, \partial D) \subset (V, \partial_+ V)$ with ∂D essential in $\partial_+ V$ and disk $(E, \partial E) \subset (W, \partial_+ W)$ with ∂E essential in $\partial_+ W$, $E \cap D = \partial D \cap \partial E \neq \emptyset$. A surface F in a 3–manifold M is incompressible if there is no disk in M with boundary an essential curve on F and interior disjoint from F.

Some key properties that follow from those established by M Scharlemann and A Thompson in [26] for a 3–manifold in thin position are the following:

(1) Every thin level is incompressible.

(2) The thin levels cut the 3–manifold into (not necessarily connected) submanifolds.

(3) Each such submanifold contains one thick level.

(4) The thick level defines a strongly irreducible Heegaard splitting on the submanifold.

5 A digression: Strongly irreducible generalized Heegaard splittings

Strongly irreducible generalized Heegaard splittings deserve to be mentioned in this context. A strongly irreducible generalized Heegaard splitting of a 3–manifold is a sequence of disjoint surfaces $S_1, F_1, \ldots, F_{k-1}, S_k$ that has the following properties:

(1) S_1 bounds a handlebody or cuts off a compression body;

(2) S_i and F_i cobound a compression body and S_i corresponds to ∂_+;

(3) F_i and S_{i+1} cobound a compression body and S_{i+1} corresponds to ∂_+;

(4) S_k bounds a handlebody or cuts off a compression body;

(5) the interiors of the aforementioned handlebodies and compression bodies are disjoint;

(6) F_i is incompressible and S_i is weakly incompressible.

(Note that S_i are strongly irreducible Heegaard surfaces for the components of M cut open along $\bigcup_{i=1}^{k} F_i$.) We emphasize that the surface above may be disconnected.

A Heegaard splitting corresponds to a handle decomposition which corresponds to a Morse function. Changing the order in which handles are attached changes this Morse function by interchanging the levels of the critical points, called a *handle slide*. Given a 3–manifold M and a Morse function f corresponding to a Heegaard splitting $M = V \cup_S W$ we may consider all Morse functions on M that differ from f only by handle slides. From the point of view here, this gives us a condition C that we impose on our Morse functions. Combining C with g, $\mathcal{O}$ and f as in the definition of thin position for 3–manifolds yields a conditional version of thin position for 3–manifolds. A manifold decomposition that is thin in this conditional sense is called an *untelescoping* of $M = V \cup_S W$. More specifically, the untelescoping, denoted by $S_1, F_1, \ldots, F_{k-1}, S_k$, is obtained by labeling the thick level surfaces by S_i and the thin level surfaces by F_i. The results in [26] still apply in this situation. It follows that $S_1, F_1, \ldots, F_{k-1}, S_k$ is a strongly irreducible generalized Heegaard splitting.

The idea of untelescoping has been used very successfully in the study of Heegaard splittings and topics related to Heegaard splitting. We have the following meta-theorem:

Meta-theorem 5.1 If a property holds for strongly irreducible Heegaard splittings, then a related property holds for for all Heegaard splittings.

Here is the idea behind this. Suppose you want to prove a certain property, let's call it X, and suppose that you can prove it for strongly irreducible Heegaard splittings. Then you should be able to prove X for essential surfaces, as these are much better behaved. Now here's what you do: you prove X for F_i, since they are essential. Then for S_i, since they are strongly irreducible Heegaard splitting (although beware—they are splittings of manifolds with boundary!). Finally you retrieve the original Heegaard splitting via a very well understood a process called amalgamation [30]. Now all that's left is to ask: *what is the related property that survives this ordeal?*

One of the first explicit applications of this meta-theorem can be seen in the following two theorems concerning tunnel numbers of knots. The tunnel number of a knot is the least number of disjoint arcs that must be drilled out of a knot complement to obtain a handlebody. A collection of such arcs is called a tunnel systems of the knot. Tunnel systems of knots correspond to Heegaard splittings.

A concept that deserves to be mentioned here is the following: A knot is *small* if its complement contains no closed essential surfaces. It follows that if a knot is small, then any tunnel system realizing the tunnel number of the knot corresponds to a strongly irreducible Heegaard splitting.

Theorem 2 (Morimoto–Schultens [15]) *If K_1, K_2 are small knots, then*

$$t(K_1 \# K_2) \geq t(K_1) + t(K_2)$$

It is easy to see that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$, so this result is quite tight. The meta-theorem was also used to bound below the degeneration of tunnel number for knots that are not necessarily small:

Theorem 3 (Scharlemann–Schultens [23])

$$t(K_1 \# K_2) \geq \tfrac{2}{5}(t(K_1) + t(K_2))$$

We now describe another application of our meta-theorem. A knot is called *m-small* if the meridian does not bound an essential surface; by Culler, Gordon, Luecke and Shalen [1, Lemma 2.0.3] all small knots in $\mathbb{S}^3$ are m-small. However, minimal tunnel systems of m-small knots do not always correspond to strongly irreducible Heegaard splittings. Let K be a knot and t its tunnel number. Denote the bridge number of a knot K with respect to a genus t Heegaard splitting by $b_1(K)$. Morimoto observed that if $b_1(K_1) = 1$ (such knots are also called $(t, 1)$ knots) then the tunnel number degenerate: $t(K_1 \# K_2) < t(K_1) + t(K_2) + 1$. He conjectured that this is a necessary and sufficient condition and proved this conjecture for m-small knots in $\mathbb{S}^3$ [14]. This was generalized by applying the meta-theorem:

Theorem 4 (Kobayashi–Rieck [13]) *Let $K_1 \subset M_1, \ldots, K_n \subset M_n$ be m-small knots. Then $t(\#_{i=1}^n K_1) < \Sigma_{i=1}^n t(K_i) + n - 1$ if and only if there exists a non-empty proper subset $I \subset \{1, \ldots, n\}$ so that $b_1(\#_{i \in I} K_i) = 1$.*

Oddly enough, the exact same meta-theorem that led to the generalization of Morimoto's Conjecture for m-small knots, also led to disproving it:

Theorem 5 (Kobayashi–Rieck [11, 12]) *There exist knots K_1, $K_2 \subset \mathbb{S}^3$ so that $b_1(K_1) > 1$ and $b_1(K_2) > 1$ but:*

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2).$$

6 Additivity properties

Widths of knots behave erratically under connected sum of knots. Progress in understanding this phenomenon is obstructed by the fact that little is known about the width of specific knots. A. Thompson was one of the first to investigate knots in thin position in their own right. A knot is called *meridionally planar small* (or mp-small) if the meridian does not bound an essential meridional surface. By definition m-small knots are mp-small. As mentioned above, by the highly technical [1, Lemma 2.0.3], small knots are m-small; thus, the family of mp-small knots contains all small knots.

Thompson proved the following theorem, see [32]:

Theorem 1 (Thompson) *If $K \subset S^3$ is mp-small then a height function h realizing the width of K has no thin levels.*

The idea of the proof is: a thin level would give a meridional planar surface. Compressing this surface yields an incompressible meridional planar surface. Some amount of work then shows that this incompressible meridional planar surface has an essential component (that is, a component that is not a boundary parallel annulus). In Section 8 we discuss generalizations of this theorem.

Thus an mp-small knot K in thin position has some number (say m) of maxima and m minima and all the maxima are above the minima. By Lemma 3.1 the width of K is exactly

$$w(K) = 2m^2.$$

Let b be the bridge number of K, that is. Clearly, $m \geq b$. On the other hand, after placing K in bridge position its width is $2b^2$, showing that $b \geq m$. We conclude that m is the bridge number. This is summarized in the following corollary which is sometimes referred to informally by saying that for mp-small knots "thin position = bridge position":

Corollary 6.2 (Thompson) *If the knot K in $\mathbb{S}^3$ is mp-small, then thin position for K is bridge position.*

The greatest challenge in this and many other investigations of thin position for knots is that thin levels need not be incompressible. This fact is used to advantage by D Heath and T Kobayashi in [5] to produce a canonical tangle decomposition of a knot and in [7] to produce a method to search for thin presentations of a knot. M Tomova has made strides in understanding this phenomenon, see [33]. We discuss these theories below. In [5], D Heath and T Kobayashi also exhibit a knot containing a meridional incompressible surface that is not realized as a thin level in a thin presentation of the knot. This propounds the idea that a decomposing sphere for a connected sum need not be realized as a thin level in a thin presentation of a composite knot.

One thing we do know concerning additivity properties of width of knots is the following:

$$w(K_1 \# K_2) \leq w(K_1) + w(K_2) - 2$$

To see this, stack a copy of K_1 in thin position on top of a copy of K_2 in thin position. The width of the connected sum is then bounded above by the relative width of the resulting presentation.

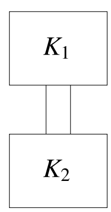

Figure 10: Connected sum of knots

A result of Y Rieck and E Sedgwick proven in [19] can be paraphrased as follows:

Theorem 3 (Rieck–Sedgwick) *If K_1, K_2 are mp-small knots, then thin position of $K_1 \# K_2$ is related to thin position of K_1, K_2 as pictured in Figure 10. In particular,*

$$w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$$

Given a presentation of $K_1 \# K_2$ in thin position and a decomposing sphere S, Y. Rieck and E. Sedgwick proceed as follows: They first show that the connected sum must have a thin level. This is accomplished as follows: For any knot K, Lemma 3.1 gives $w(K) \leq 2b(K)^2$, where $b(K)$ is the bridge number of K. A result of Schubert [28], states that the bridge number of knots is subadditive, that is,

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1.$$

A standard computation shows that the function

$$f(x, y) = xy - x - y + 1$$

is strictly greater than 0 for $x, y \geq 2$. Thus since bridge number is always at least 2,

$$w(K_1) + w(K_2) - 2 \leq 2b(K_1)^2 + 2b(K_2)^2 - 2 < 2(b(K_1) + b(K_2) - 1)^2.$$

Hence thin position can't be bridge position for $K_1 \# K_2$, there must be a thin level.

Their next steps are more technical: They show that for any decomposing annulus in the complement of $K_1 \# K_2$, a spanning arc can be isotoped into a thin level. Finally, they show that a thin level containing the spanning arc of a decomposing annulus must in fact be a decomposing annulus. This establishes their result.

Note that the application of Schubert's Theorem above shows that for any knot K_1 and K_2, after placing $K_1 \# K_2$ in thin position a bridge position is not obtained, in the sense that there is a thin sphere. However, counting the number of maxima in Figure 10 shows that:

Corollary 6.4 (Rieck–Sedgwick) *If K_1 and K_2 are mp-small knots, then the number of maxima for $K_1 \# K_2$ in thin position is the bridge number of $K_1 \# K_2$.*

Examples of Scharlemann and Thompson [27] suggest that this is not always the case.

The proofs in Schubert [28], Schultens [31] and Rieck–Sedgwick [19] do not carry over to knots in general. To give some idea of the complexity of the situation, we illustrate the problems with the strategy in [31]. Rather than working with decomposing spheres and annuli, that strategy employs swallow-follow tori. See Figure 11.

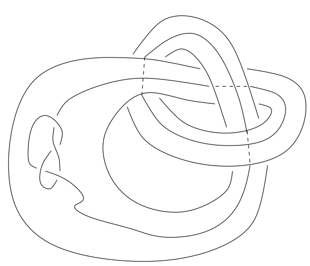

Figure 11: A swallow-follow torus

Given $K_1 \# K_2$ and a decomposing sphere S, consider a collar neighborhood of $(K_1 \# K_2) \cup S$ in $\mathbb{S}^3$. Its boundary consists of two tori. A torus isotopic to either of these tori is called a swallow-follow torus. Figure 11 illustrates the case in which K_1 is a figure eight knot and K_2 is a trefoil.

Swallow-follow tori often prove more effective in studying connected sums of knots, in large part because they are closed surfaces. The argument in [31] fails in settings where the swallow-follow torus is too convoluted. See Figure 12.

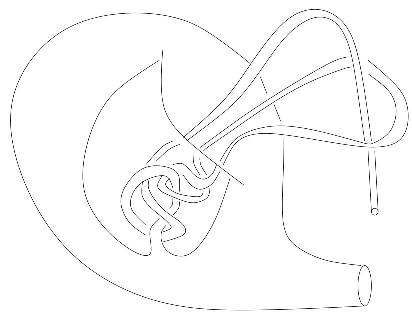

Figure 12: Convoluted portion of a swallow follow torus

The philosophical correspondence between tunnel numbers and strongly irreducible generalized Heegaard splittings on the one hand and bridge position and thin position of knots deserves to be investigated more closely. Suffice it to say that this correspondence played a role in the discovery of the argument yielding the inequality below. The lack of degeneracy for tunnel numbers of small knots is mirrored by their lack of degeneracy of width. Nevertheless, more generally, tunnel numbers do degenerate under connected sum and so might their widths. M Scharlemann and A Thompson conjecture that there are knots whose width remains constant under connected sum with a 2–bridge knot. See [27].

Finally, a lower bound on the width of the connected sum in terms of the widths of the summands was established by Scharlemann and Schultens:

Theorem 5 (Scharlemann–Schultens [24]) *For any two knots K_1, K_2,*

$$w(K_1 \# K_2) \geq max(w(K_1), w(K_2))$$

Corollary 6.6 (Scharlemann–Schultens [24]) *For any two knots K_1, K_2,*

$$w(K_1 \# K_2) \geq \frac{1}{2}(w(K_1) + w(K_2))$$

The fact that width of 3–manifold behaves well from many points of view has initiated reconsiderations of the notion of thin position for knots. One is tempted to redefine the notion of thin position for knots so as to avoid the difficulties it engenders. Scharlemann and Thompson have defined a notion of "slender knots" which lies outside of the scope of this article.

7 Work of Heath and Kobayashi

D J Heath and T Kobayashi were the first to use the possible compressibility of thin levels to advantage. They made great strides in understanding many issues related to thin position of knots. In this section we briefly summarize their results. Details on these results may be found in three of their joint papers [5, 6, 7]. The illustrations alone are each worth a thousand words. Our brief summary requires a number of definitions. Some of these are analogous to other definitions in this survey, but are given here in a slightly different context.

Given a link L in $\mathbb{S}^3$, our height function, $h(x)$, can be thought of as resulting from looking at $\mathbb{S}^3 - 2\text{ points } = \mathbb{S}^2 \times \mathbb{R}$. This restriction of $h(x)$ to $\mathbb{S}^2 \times \mathbb{R}$ is then simply projection onto the $\mathbb{R}$ factor. We let $p(x)$ be the projection onto the $\mathbb{S}^2$ factor. Consider a meridional 2–sphere S, that is, a 2–sphere in $\mathbb{S}^3$ that intersects L in points. (In the complement of L, the remnant of S has boundary consisting of meridians.)

The 2–sphere S is said to be *bowl like* if all of the following hold (see Figure 13):

(1) $S = F_1 \cup F_2$ and $F_1 \cap F_2 = \partial F_1 = \partial F_2$;

(2) F_1 is a 2–disc contained in a level plane;

(3) $h|_{F_2}$ is a Morse function with exactly one maximum or minimum;

(4) $p(F_1) = p(F_2)$;

(5) $p|_{F_2} : F_2 \to p(F_2)$ is a homeomorphism;

(6) all points of intersection with L lie in F_1.

A bowl like 2–sphere is *flat face up* (*flat face down*) if F_1 is above (below) F_2 with respect to h.

Let $F = h^{-1}(r)$, for some regular value r of h, be a thick 2–sphere for L. Let N_0 (N_1) be a thin 2–sphere lying directly above (below) F. Let D be a disc such that $\partial D = \alpha \cup \beta$ with α a subarc of L containing a single critical point which is a maximum, and $\beta = D \cap F$. Similarly, let D' be a disc such that $\partial D' = \alpha' \cup \beta'$ with α' a subarc of L containing a single critical point which is a minimum, and $\beta' = D' \cap F$. Assume that the interiors of β and β' are disjoint and that $\alpha \cup \alpha'$ is not a complete component of L. Then D and D' are called *a bad pair of discs*. They are called *a strongly bad disc pair* if $D \cap N_0 = \emptyset = D' \cap N_1$.

Let $S = S_1 \cup \cdots \cup S_n$ be a collection of bowl like 2–spheres for a link L and $C_0, \ldots C_n$ be the closure of the components of $S^3 - S$. Note that each S_i separates $\mathbb{S}^3$ into two sides. The one not containing the 2 points that have been removed from $\mathbb{S}^3$ is considered

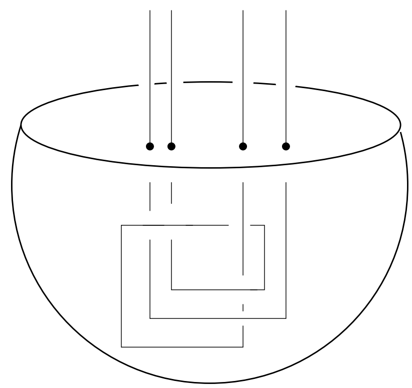

Figure 13: A bowl like 2–sphere (flat face up)

to lie inside S_i. For $i = 1, \ldots, n$, C_i is a punctured copy of the 3–ball lying inside S_i. Furthermore, C_0 is the component which does not lie interior to any S_i, and C_i ($i = 1, ..., m$) is the component lying directly inside of S_i.

Let $L_i = L \cap C_i$. We define thin (thick) level disk analogously to thin (thick) level spheres. We say that $L_i, i \neq 0$, is in *bridge position* if there exists some thick 2–disk $D_i \subset C_i$ for L_i such that all maxima (minima) of L_i are above (below) D_i, and every flat face down (up) bowl like 2–sphere S_j contained in the "inner boundary" of C_i (where C_i meets S_j for $j \neq i$) is above (below) D_i. We say that L_0 *is in bridge position* if there exists some thick 2–sphere $D_0 \subset C_0$ for L_0 having the analogous properties. Finally, let L' be a portion of the link L lying inside the bowl like 2–sphere S. We also say that L' *is in bridge position* if there exists some thick 2–disk D for L' such that all maxima (minima) of L' are above (below) D.

We wish to associate a graph with the above information. To this end we will suppose that, given S and $C_0, \ldots, C_n$ as above, the following properties are satisfied. If they are, the system is said to enjoy *Property 1*.

(1) For each C_j, ($j = 0, 1, ..., m$), we have one of the following:
   (a) there are both a maximum and a minimum of L in C_j; or
   (b) there does not exist a critical point of L in C_j.
(2) There exists a level 2–sphere F_0 in C_0 such that both of the following hold:
   (a) every flat face down (up, respectively) bowl like 2–sphere in ∂C_0 lies above (below, respectively) F_0; and

(b) every maximum (minimum, respectively) of L in C_0 (if one exists) lies above (below, respectively) F_0, and it is lower (higher, respectively) than the flat face down (up, respectively) bowl like 2–spheres in ∂C_0.

(3) For each i, $(i = 1, ..., m)$, there exists a level disk F_i properly embedded in C_i such that both of the following hold:

(a) every flat face down (up, respectively) bowl like 2–sphere in ∂C_i lies above (below, respectively) F_i , and

(b) every maximum (minimum, respectively) of L in C_i (if one exists) lies above (below, respectively) F_i, and it is lower (higher, respectively) than the face down (up, respectively) bowl like 2–spheres in $\partial C_i - S_i$.

A *spatial graph* G is a 1–complex embedded in the 3–sphere. G is a *signed vertex graph* if each vertex of G is labeled with either a $+$ or a $-$. The *width of* G is defined as follows. Suppose that the vertices of G labeled with $+$ ($-$, respectively) have the same height and are higher (lower, respectively) than any other point in G. Suppose further that $h|_{G-\{vertices\}}$ is a Morse function. We say that G is in *bridge position* if each maximum in $G - \{vertices\}$ is higher than any minimum of $G-\{vertices\}$. In general, let $r_1, \ldots, r_{n-1}$ $(r_1 < \cdots < r_{n-1})$ be regular values between the critical values in $G - \{vertices\}$. Then define the width of G to be the following $w(G) = \Sigma_{i=1}^{n-1}|G \cap h^{-1}(r_i)|$. For a signed vertex graph in bridge position *the bridge number* is $|F \cap G|/2$ where F is a level 2–sphere such that every maximum of G is above F and every minimum of G is below F. The minimum of the bridge numbers for all possible bridge positions of G is the *bridge index* of G.

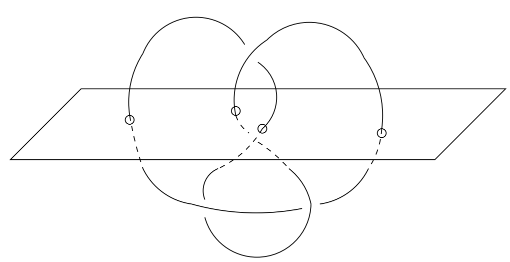

Figure 14: The Figure 8 knot in bridge position

Let L, C_j $(j = 0, 1, ..., m)$ be as above. We can obtain a signed vertex graph G_j from $(C_j, L \cap C_j)$ as follows: In the case that $j = 0$, shrink each component of ∂C_0 to a vertex. Then pull up (down, respectively) the vertices obtained from flat face down (flat face up, respectively) 2–spheres so that they lie in the same level. We obtain the signed vertex graph G_0 by assigning $+$ to the former and $-$ to the latter. By (2) of Property 1, we see that G_0 is in a bridge position.

Suppose that $j \neq 0$. In this case we may deform C_j by an ambient isotopy, f_t, of $\mathbb{S}^3$ which does not alter the flat face of S_j so that $f_1(C_j)$ appears to be of type C_0. This isotopy moves infinity "into" C_j. For details see for instance the "Popover Lemma" in [31]. Then we apply the above argument to $(f_1(C_j), f_1(L \cap C_j))$ and obtain a signed vertex graph G_j in bridge position. We say that G_j $(j = 0, 1, ..., m)$ is a *signed vertex graph associated to* S. In this process we reversed S_j and made no other changes; thus the resulting signed graph is the same as the signed graph in the case $j = 0$ but the sign of the vertex corresponding to S_j is reversed.

Let L, C_j $(j = 0, 1, ..., m)$ be as above. Then we can take a convex 3–ball R_j in the interior of C_j such that each component of $(L \cap C_j) - R_j$ is a monotonic arc connecting R_j and a component of ∂C_j, and such that

(1) R_0 lies below (above, respectively) the flat face down (up, respectively) bowl like 2–spheres in ∂C_0;

(2) R_i $(i = 1, ..., m)$ lies below (above respectively) the flat face down (up respectively) bowl like 2–spheres in $\partial C_i - S_i$.

We call R_j a *cocoon* of L associated to S.

7.1 A search method for thin position of links

Let L be a link of bridge index n and suppose that there is a list of all those meridional, essential, mutually non parallel planar surfaces in the exterior of L, that have at most $2n - 2$ boundary components. Let $S = \bigcup_{i=1}^{m} S_i$ be a union of 2–spheres in S^3 as above. Then we can obtain a number of systems of signed vertex graphs as follows: For each i, $(i = 1, ..., m)$, we assign $+$ to one side of S_i and $-$ to the other. Note that there are 2^m ways to make such assignments. Let $C_0, C_1, \ldots, C_m$ be as above. Then for each j, $(j = 0, 1, ..., m)$ the collar of each component of ∂C_j is assigned either a $+$ or a $-$. By regarding each component of ∂C_j as a very tiny 2–sphere, we obtain a signed vertex graph, say G_j, from $L \cap C_j$.

Now we assume, additionally, that we know the bridge indices of all the signed vertex graphs obtained in this manner. Then, for each system of signed vertex graphs, we take minimal bridge presentations, say $G_0, G_1, ..., G_m$, of the signed vertex graphs. We expand the vertices of $G_0, G_1, ..., G_m$ to make + vertices (- vertices, respectively) flat face down (up, respectively) bowl like 2–spheres. Then we combine the pieces, applying the inverse of deformations to obtain a position of L, say L', and a union of bowl like 2–spheres S' with respect to which L' satisfies Property 1 above. Let $R_0, R_1, ..., R_m$ be the cocoons of L' associated to S'. Then consider all possible orders on $\{R_0, R_1, ..., R_m\}$ which are compatible with relative positions in L'. All such orders are realized as a position of L.

Theorem 1 (Heath–Kobayashi [7, Theorem 2]) *There is a thin position of L that is realized through the process described above.*

7.2 Essential tangle decomposition from thin position of a link

We say that two links L and L' in $\mathbb{S}^3$ are *h-equivalent* if there exists an ambient isotopy, f_t, such that $f_0(L) = L$, $f_1(L) = L'$ and such that for every $x \in L$ we have $h(f_1(x)) = h(x)$.

Proposition 7.2 (Heath–Kobayashi [5, Proposition 3.7]) *If a link L has the property that thin position differs from bridge position, then there exists an ambient isotopy f_s, such that $L' = f_1(L)$ is h-equivalent to L and L' has a tangle decomposition by a finite number of non-trivial, non-nested, flat face up, bowl like 2–spheres, each of which is incompressible in the link complement. In this decomposition we have a tangle "on top" (above P) with all of the incompressible 2–spheres below it connected by vertical strands.*

Theorem 3 (Heath–Kobayashi [5, Theorem 4.3]) *Let L be a link in thin position, and S as above. Then there exists an ambient isotopy for L to a link L' so that there exists a collection of incompressible bowl like 2–spheres S' for L' such that there is a one to one correspondence between the components of $\mathbb{S}^3 - S'$ that contain maximum (and minimum) of L' and the components of $\mathbb{S}^3 - S$ that contain maximum (and minimum) of L.*

7.3 Locally thin position for a link

Perhaps the greatest weakness of thin position, as with many knot invariants that are defined in terms of a global minimum, is that it is hard to determine. On the other hand, many of its applications rely only on local properties of thin position. In order to address this issue, Heath and Kobayashi define a local version of thin position.

Definition 7.4 A link L is said to be in local thin position if it satisfies the following two properties with respect to the height function h:

(1) no thick 2–sphere for L has a strongly bad pair of discs, and

(2) There exists a decomposition of L with bowl like 2–spheres $S_1, \dots, S_n$ such that each S_i is incompressible and ∂–incompressible, and so that L is in bridge position in the complement of $\cup S_i$.

They then prove the following main result and two corollaries:

Theorem 5 (Heath–Kobayashi [6, Main Theorem 3.1]) *Every non-splittable link has a locally thin presentation.*

Corollary 7.6 (Heath–Kobayashi [6, Corollary 3.4]) *Any locally thin position of the unknot is trivial.*

Corollary 7.7 (Heath–Kobayashi [6, Corollary 3.5]) *Any locally thin position of a 2–bridge knot is in 2–bridge position.*

These corollaries both show the strength and the weakness of local thin position. It is not as easy to compute as one might hope or this would mean that recognizing the unknot and 2–bridge knots would be easy, but in exchange it contains significant information when it is computed.

8 Compressibility of thin levels

Ying-Qing Wu began an investigation of the thin levels for knots in thin position. He proved the following about the thinnest thin level, that is, the thin level that meets the knot in the fewest number of points.

Theorem 1 (Wu [34]) *If K is in thin position with respect to h, then the thinnest thin level of K is incompressible.*

Wu's strategy is to show that if a thin level is compressible, then the surface obtained by compressing it is parallel to another thin level. His result then follows by induction. He also demonstrates applications of this result: He uses it to give an alternative proof of the Rieck–Sedgwick Theorem.

Maggy Tomova continued this investigation in [33]. She proved more refined results about compressing disks for thin levels of links in thin position. Her results rely on a number of concepts, observations and lemmas. We give a very brief overview, for details see [33]. In particular, note that the description below relies on many technical lemmas.

Suppose the link L in $\mathbb{S}^3$ is in thin position. Further suppose that $P = h^{-1}(r)$ is a thin level for K and that D is a compressing disk for P in the complement of K. We may assume that the interior of D lies entirely above or entirely below P, say, the former. To help our visualization of the situation, we imagine D as a cylinder lying vertically over ∂D and capped off with the maximum, ∞, of h. It is then clear that D partitions the portion of L lying above P into two subsets. Denote the portions of L above P that are separated by D by α and β. See Figure 15.

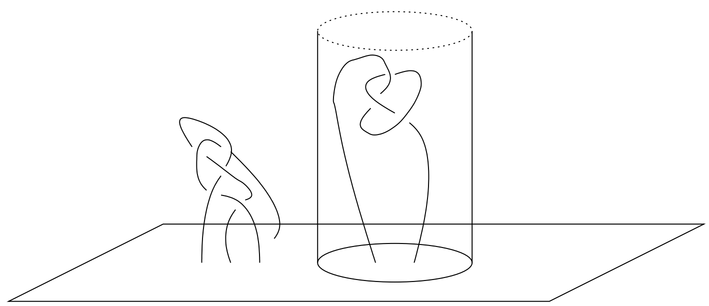

Figure 15: The portions α and β of L

Now play off α versus β. An alternating thin level is a thin level $P' = h^{-1}(r')$ above P such that the first minimum above P' lies on α and the first maximum below P' lies on β or vice versa. As it turns out, alternating thin levels necessarily exist; furthermore, for any adjacent alternating thin levels, either the portion of α or the portion of β lying between the two alternating thin levels is a product.

Interestingly, if we number the alternating thin levels above P by $A_1, \ldots, A_n$, such that $h(A_{j-1}) < h(A_j)$, then the sequence $w_1, \ldots, w_n$ defined by $w_j = \#|K \cap A_j|$ is strictly decreasing. The class of alternating thin levels can be enlarged to include other thin levels that satisfy certain technical properties enjoyed by alternating thin levels. The resulting class of surfaces are the potentially alternating surfaces. Compressing disks such as D can then be assigned a height: Assign D the height k if $D \cap A_{k-1} \neq \emptyset$ but $D \cap A_k = \emptyset$.

The short ball for a compressing disk D for P is the ball bounded by D and a subdisk of P that contains the shorter of α or β, that is, that portion of the knot whose absolute maximum is lower than that of the other. (By transversality, these two maxima do not lie on the same level.) The compressing disk D for P is reducible if there is a disk E whose interior lies in the short ball for D, whose boundary is partitioned into an arc τ on D and an arc ω on P and for which ω is essential in $P - (\partial D \cup L)$. A compressing disk is irreducible if it is not reducible.

A key result is the following:

Theorem 2 (Tomova [33]) *Suppose D and D' are two irreducible compressing disks for P, and α, α' are the strands of L lying in the corresponding short balls. Then* $\mathrm{height}(D) = \mathrm{height}(D')$ *implies $\alpha = \alpha'$. Otherwise, $\alpha \cap \alpha' = \emptyset$.*

Corollary 8.3 (Tomova [33]) *Any two distinct irreducible compressing disks for P of the same height must intersect.*

See Figure 16.

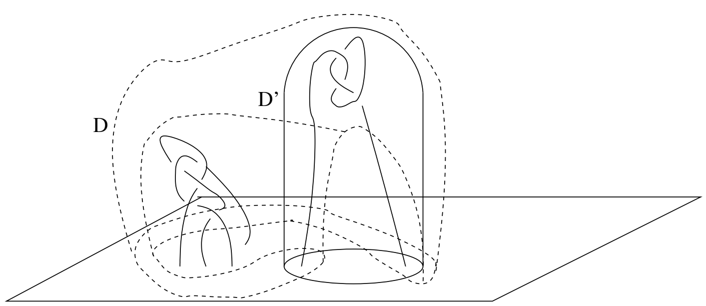

Figure 16: Two disks of the same height

Theorem 4 (Tomova [33]) *There exists a collection of disjoint irreducible compressing disks for P that contains one representative from each possible height.*

In certain situations, these results suffice to guarantee unique compressing disks for thin levels!

9 2–fold branched covers

Let K be a knot in $\mathbb{S}^3$ and let M be the 2–fold branched cover of $\mathbb{S}^3$ over K. It is natural to ask the following question: How is thin position of K related to thin position of M?

This question is investigated by Howards and Schultens [9]. (A related question about the behavior of the Heegaard genus under double covers was investigated by Rieck and Rubinstein [17]).

A height function on $\mathbb{S}^3$ lifts to a Morse function on M. The thick and thin levels of K and M are related as follows: Denote the thick levels of K by $S_1, \ldots, S_n$ and the thin levels by $L_1, \ldots, L_{n-1}$. Here the S_i's and L_i's are spheres that meet the knot some (even) number of times. In fact, each S_i meets K at least 4 times and each L_i meets K at least 2 times. Denote the surface in M corresponding to S_i by $\tilde{S}_i$ and the surface in M corresponding to L_i by $\tilde{L}_i$. If S_i meets K exactly $2l$ times, then $\tilde{S}_i$ is a closed orientable surface of genus $l-1$. And if L_i meets K exactly $2l$ times, then $\tilde{L}_i$ is a closed orientable surface of genus $l-1$.

Compressing disks for $\tilde{S}_i$ may be constructed by taking a disk D in $\mathbb{S}^3$ that is disjoint from $L_1, \ldots, L_{n-1}$, whose interior is disjoint from $S_1, \ldots, S_n$ and whose boundary is partitioned into an arc a in S_i and an arc b in K that has exactly one critical point. (Such a disk is called a strict upper/lower disk.) The 2–fold branched cover $\tilde{D}$ of D has its boundary on $\tilde{S}_i$ and is a compressing disk for $\tilde{S}_i$. This illustrates the fact that $\tilde{L}_{i-1}$ and $\tilde{S}_i$ and also $\tilde{L}_i$ and $\tilde{S}_i$ cobound compression bodies.

Now if K is in thin position, then one may ask whether or not the manifold decomposition that M inherits is in thin position.

Theorem 1 (Howards–Schultens) *If K is a 2–bridge knot or a 3–bridge knot, then the manifold decomposition that M inherits is in thin position.*

This result is not true for knots in general. Consider for instance torus knots. For torus knots the manifold decomposition that their 2–fold branched cover inherits is not necessarily in thin position. To see this, consider the following: The complement of a torus knot is a Seifert fibered space fibered over the disk with two exceptional fibers. This places restrictions on the type of incompressible surfaces that can exist. In particular, it rules out meridional surfaces. For a discussion of incompressible surfaces in Seifert fibered spaces, see for instance Hempel [8] or Jaco [10]. It follows that K is mp-small.

Now Thompson's Theorem (Theorem 1) implies that thin position for K is bridge position. Bridge numbers for torus knots can be arbitrarily large. Specifically, if K is a (p,q)–torus knot, then the bridge number of K is $\min\{p,q\}$. This was proved by Schubert in [28]. For a more contemporary and self contained proof see Schultens [29]. Thus for K in thin position, the manifold decomposition that the 2–fold branched cover inherits is a Heegaard splitting of genus $\frac{\min\{p,q\}}{2} - 1$.

On the other hand, the 2–fold branched cover of $\mathbb{S}^3$ over a torus knot is a small Seifert fibered space. Specifically, the 2–fold branched cover of $\mathbb{S}^3$ over the (p,q)–torus knot is a Seifert fibered space fibered over $\mathbb{S}^2$ with three exceptional fibers of orders $p, q, 2$. (Such manifolds are also called Brieskorn manifolds.) But any such manifold possesses Heegaard splittings of genus 2.

10 Questions

The following questions deserve to be considered:

(1) Develop a less unwieldy notion of thin position for knots.

(2) Find an algorithm to detect the width of a knot. In light of the discussion at the end of Section 7 we ask: find an algorithm to place a knot in local thin position for knots.

(3) Characterize the compressibility of thin levels for knots in thin position.

(4) Construct knots of arbitrarily large width.

(5) Apply the concept of thin position in completely different settings.

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Received: 11 March 2006 Revised: 20 January 2007