1 Required problems

1. Prove that the relation $R$ given below is an equivalence relation on the set $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ and then find the distinct equivalence classes for $R$.

   \[ mRn \iff 5 \mid (m^2 - n^2) \]

2. Prove that the relation $V$ given below is an equivalence relation on the set $\mathbb{R}$ of all real numbers and then find the distinct equivalence classes for $V$.

   \[ xVy \iff |x| = |y| \]

3. Suppose that a fair coin is tossed three times and the side showing face up is noted. Let $HHT$ indicate the outcome heads on the first two tosses and tails on the third, $THT$ the outcome tails on the first and third tosses and heads on the second, and so forth.

   (a) List the eight elements in the sample space whose outcomes are all the possible head-tail sequences obtained in the three tosses.
   (b) What is the probability that exactly one toss results in a head?
   (c) What is the probability that at least two tosses result in a head?
   (d) What is the probability that no head is obtained?

4. In baseball’s World Series, the first team to win four games wins the series.

   (a) Suppose team $A$ wins the first two games. How many ways can the series be completed? (Draw a tree.)
   (b) How many ways can a World Series be played if no team wins two games in a row?

5. A combination lock requires three selections of numbers, each from 1 through 30.

   (a) How many different combinations are possible?
   (b) Suppose the locks are constructed in such a way that no number may be used twice. How many different combinations are possible?

6. Prove that for all integers $n \geq 2$,

   \[ P(n + 1, 3) - P(n, 3) = 3P(n, 2). \]
7. Six people attend the theater together and sit in a row with exactly six seats.

(a) Suppose one of the six is a doctor who must sit on an aisle in case she is paged. How many ways can the people be seated together in the row with the doctor on an aisle? (Assume there are two aisles, one on each side.)

(b) Suppose the six people consist of three heterosexual couples and each couple wants to sit together with the husband on the left. How many ways can the six be seated together in the row?

8. What is the probability that a five-digit integer chosen at random is divisible by 5? (A five-digit integer is an integer between 10,000 and 99,999.)

9. How many integers from 1 through 1,000 are neither multiples of 2 nor multiples of 9?

10. A computer programming team has 13 members.

(a) How many ways can a group of 7 be chosen to work on a project?

(b) Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?

(c) Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

2 Bonus Problems

Please hand in on a sheet separate from your homework.

1. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ where $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $k_1, k_2, \ldots, k_m$ are positive integers. How many ways can $n$ be written as a product of two positive integers that have no common factors

(a) assuming that order matters (i.e. $8 \cdot 15$ and $15 \cdot 8$ are regarded as different)?

(b) assuming that order does not matter (i.e. $8 \cdot 15$ and $15 \cdot 8$ are regarded as the same)?

2. How many permutations of $abcde$ are there in which the first character is $a, b,$ or $c$ and the last character is $c, d,$ or $e$?