MATH117 Homework 3: due Friday, 19 September

Remember to justify your answers!

(1) Determine whether the property below is true for all integers, true for no integers, or true
for some integers and false for other integers.

\[ 3n^2 - 4n + 1 \text{ is prime.} \]

(2) Prove that for all integers \( n \), if \( n \) is odd then \( n^2 \) is odd.

(3) For each of the following statements, determine whether it is true or false. Justify your
answer with a proof or a counterexample, as appropriate.

(a) For all integers \( m \), if \( m > 2 \) then \( m^2 - 4 \) is composite.

(b) The difference of the squares of any two consecutive integers is odd.

(4) Prove that if one solution for a quadratic equation of the form \( x^2 + bx + c = 0 \) is rational
(where \( b \) and \( c \) are rational), then the other solution is also rational. (Use the fact that if
the solutions of the equation are \( r \) and \( s \), then \( x^2 + bx + c = (x - r)(x - s) \).)

(5) Prove (directly from the definition of divisibility) that for all integers \( a, b, \) and \( c \), if \( a|b \) and
\( a|c \) then \( a|(b - c) \).

(6) Determine whether the following statement is true or false. If it is true, prove it. If it is
false, provide a counterexample.

For all integers \( a \) and \( n \), if \( a|n^2 \) and \( a \leq n \), then \( a|n \).

(7) Suppose that in standard form \( a = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k} \), where \( k \) is a positive integer, \( p_1, p_2, \ldots, p_k \)
are prime numbers, and \( e_1, e_2, \ldots e_k \) are positive integers.

(a) What is the prime factorization for \( a^3 \)?

(b) Find the least positive integer \( k \) such that \( 2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k \) is a perfect cube (i.e. equals
an integer to the third power). Write the resulting product as a perfect cube.