1 Introduction

Symmetric functions are vital to the study of combinatorics because they provide valuable information about partitions and permutations, topics which constitute the core of the subject. The significance of symmetric function theory is manifest by its connections to other branches of mathematics, including group theory, representation theory, Lie algebras, and algebraic geometry.

One important basis for the vector space of symmetric functions is the Schur function basis, which has an elegant combinatorial construction. Many other symmetric function bases are special cases of the Schur functions. Schur functions arise in representation theory as the characters of the irreducible representations of the general linear group. They provide insight into the multiplicative structure of the cohomology ring of the Grassmannian. Combinatorial properties of the Schur functions, such as the Littlewood-Richardson Rule and RSK algorithm, provide powerful tools for attacking problems in combinatorics such as permutation enumeration and plane partitions.

Macdonald polynomials are a generalization of symmetric functions which have recently generated a significant amount of excitement. The combinatorial formulas for Macdonald polynomials and nonsymmetric Macdonald polynomials [4], [6] permit a combinatorial description of nonsymmetric functions, called Demazure atoms, which decompose the Schur functions. We explore several properties of these functions and their applications to representation theory [14], [15].

Quasisymmetric functions are objects that bridge the gap between nonsymmetric functions and symmetric functions. In addition to providing combinatorial information about symmetric functions, the rich structure of quasisymmetric functions relates to other algebraic structures as well. For instance, the Hopf algebra of quasisymmetric functions is dual to the Hopf algebra of noncommutative symmetric functions, which appears as the Solomon descent algebra. We introduce a new basis for quasisymmetric functions which decompose the Schur functions and we describe several properties they share with Schur functions [7], [8]. We call these functions quasisymmetric Schur functions because of their close relationship to the Schur functions.

2 Demazure atoms

A symmetric function of degree $n$ is a formal power series $f(x) = \Sigma_{\alpha}c_{\alpha}x^{\alpha}$ where $c_{\alpha}$ is a rational number, where $\alpha$ ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, ...)$ of $n$, where $x^{\alpha} = \prod x_{i}^{\alpha_i}$, and where $f(x_{\sigma(1)}, x_{\sigma(2)}, ...) = f(x_1, x_2, ...)$ for every permutation $\sigma$ of the positive integers. (We consider a weak (resp. strong) composition of $n$ to be an sequence of non-negative (resp. positive) integers which sum to $n$.) The set of all symmetric functions of a given degree forms a vector space over the rational numbers. A central theme in the
theory of symmetric functions is the description of bases for this vector space and their relationships and properties [18].

The Schur functions, one of the most useful bases for symmetric functions, can be defined combinatorially using semi-standard Young tableaux (SSYT). A partition of an integer \( n \) is a weakly decreasing sequence of positive integers which sum to \( n \). If \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a partition, its Ferrers diagram consists of rows of boxes, or cells, where the \( i \)th row contains \( \lambda_i \) cells. When these cells are filled with positive integers so that the rows are weakly increasing and the columns are strictly increasing, the resulting diagram is called a semi-standard Young tableau. (See Figure 1(a).) The Schur function \( s_\lambda \) is the formal power series \( s_\lambda(x) = \sum T x^T \), where the sum is over all semi-standard Young tableaux \( T \) of shape \( \lambda \) and weight of \( x^T \). Certain operators act on Schur functions to produce results about partitions, symmetric functions, and matrices.

The Macdonald polynomials are a special class of symmetric functions which contain a vast array of information. Macdonald [12] introduced them and conjectured that their expansion in terms of Schur polynomials should have positive coefficients. Haglund, Haiman, and Loehr found a combinatorial formula for the Macdonald polynomials which utilizes statistics on fillings of partition diagrams in 2005 [4].

Building on this work, Haglund conjectured a combinatorial formula for the nonsymmetric Macdonald polynomials. A specialization of this formula provides a set of objects that decompose the Schur functions into nonsymmetric functions indexed by compositions of \( n \) instead of partitions of \( n \). Each weak composition \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \) is associated to an augmented diagram consisting of \( \gamma_i \) cells in the \( i \)th column. The cells are then filled with positive integers according to conditions generalizing those described in [5] to obtain a semi-standard augmented filling (SSAF) (See Figure 1(b)). The weighted sums of these objects are called the Demazure atoms, \( A_\gamma \).

A composition \( \gamma \) of \( n \) is called a rearrangement of a partition \( \lambda \) of \( n \) if it consists of \( n \) parts such that when the parts are arranged in decreasing order, the \( i \)th part equals \( \lambda_i \), for all \( i \). Haglund’s formula implies that the sum of the Demazure atoms over all rearrangements of a given partition \( \lambda \) is equal to the Schur function \( s_\lambda \). We exhibit a weight-preserving bijection between SSYT and SSAF to prove the following.

**Theorem 1 (M) [14]** The sum of the Demazure atoms over all rearrangements, \( \gamma \), of \( \lambda \) is equal to the the Schur function \( s_\lambda \). Symbolically,

\[
\sum_{\gamma \vdash \lambda} A_\gamma(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n).
\]
This result gave evidence that Haglund’s conjectured formula for the nonsymmetric Macdonald polynomials was correct. Haglund, Haiman, and Loehr recently proved this conjecture [6]. Their theorem provides an algebraic proof of Theorem 1.

2.1 Properties of Demazure atoms

The combinatorial proof of Theorem 1 provides a dictionary between the symmetric and the nonsymmetric, allowing one to translate Schur function properties into Demazure atom properties. For example, the Robinson-Schensted-Knuth (RSK) algorithm [17] has an analogue in the nonsymmetric setting.

**Theorem 2 (M) [14]** There exists a bijection between $\mathbb{N}$-matrices of finite support and ordered pairs $(F,T)$ of semi-standard augmented fillings which rearrange the same shape.

The fundamental operation in the bijection of Theorem 2 is an insertion process similar to the insertion process in the ordinary RSK Algorithm. This provides a method for inserting a value into a semi-standard augmented filling and obtaining a new semi-standard augmented filling whose shape has one additional cell. The bijection commutes with the ordinary RSK algorithm and therefore retains many of its properties, including symmetry and the equality of the insertion tableaux for Knuth equivalent words. We use the bijection to determine analogues of other properties in joint work with Jeff Remmel (University of California, San Diego) and Jim Haglund (University of Pennsylvania) [9]. An undergraduate honors student has made progress on related problems.

One main research goal is to use this bijection to determine nonsymmetric analogues for other theorems and properties associated with the Schur functions. We have done this in the case of the Littlewood-Richardson multiplication rule in joint work with Jim Haglund, Kurt Luoto (University of Washington), and Steph van Willigenburg (University of British Columbia) and have found a way to ‘skew’ Demazure atoms. The classical Littlewood-Richardson rule provides a combinatorial method for computing the product of two Schur functions in the Schur function basis. We define an object called a Littlewood-Richardson skyline (LRS) to obtain the following refinement of the Littlewood-Richardson rule, which provides a method for expanding the product of a Schur function and a Demazure atom in terms of the basis of Demazure atoms.

**Theorem 3 (H-L-M-vW) [8]** Let $\lambda$ be a partition and $\gamma, \delta$ be weak compositions. In the expansion

$$A_{\gamma} \cdot s_{\lambda} = \sum_{\delta} a_{\gamma,\lambda}^{\delta} A_{\delta},$$

the coefficient $a_{\gamma,\lambda}^{\delta}$ is given by the number of Littlewood-Richardson Skylines of shape $\delta/\gamma$.

A skew tableau, $\lambda/\mu$, is obtained from two partitions $\mu \subseteq \lambda$ by removing the cells of $\mu$ from $\lambda$. The remaining cells are filled with positive integers so that the row entries are weakly increasing and the column entries are strictly increasing to obtain a skew SSYT. The skew Schur function $s_{\lambda/\mu}$ is the function generated by all skew semi-standard Young tableaux of shape $\lambda/\mu$. We define a skew SSAF in an analogous way and prove that the functions obtained from these diagrams refine the skew Schur functions. We generalize this
idea to Macdonald polynomials in the hopes of providing a combinatorial formula for the skew Macdonald polynomials appearing in the algebro-geometric setting. We conjecture that our polynomials are in fact equal to the skew Macdonald polynomials.

3 Quasisymmetric Schur functions

A quasisymmetric function is a formal power series $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ for all compositions $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and all $i_1 < i_2 < \ldots < i_k$. The fundamental quasisymmetric functions were introduced by Gessel [3] to investigate P-partitions and properties of partially ordered sets.

We introduce a new basis for quasisymmetric functions obtained by summing the Demazure atoms indexed by weak compositions which collapse to the same strong composition. We call the resulting collection of functions quasisymmetric Schur functions since they decompose the Schur functions in a natural way and share many of the properties of Schur functions [7].

In particular, we introduce a collection of diagrams called LRCs to describe a combinatorial rule for the multiplication of a Schur function and a quasisymmetric Schur function [8]. This rule contains the classical Littlewood-Richardson rule for the product of two arbitrary Schur functions since the Schur functions are sums of quasisymmetric Schur functions.

**Theorem 4** (H-L-M-vW) [8] Let $\lambda$ be a partition and $\alpha, \beta$ be strong compositions. In the expansion

$$S_\alpha \cdot s_\lambda = \sum \beta C^\beta_{\alpha\lambda} S_\beta,$$

(2)

the coefficient $C^\beta_{\alpha\lambda}$ is the number of LRCs of shape $\beta/\lambda$ whose shape collapses to $\alpha$.

When the product of two quasisymmetric Schur functions is expanded in the quasisymmetric Schur function basis, the coefficients appear to be integers. We hope to determine a combinatorial formula for these coefficients and to define a Hopf algebra structure on this basis. Experimental data suggests that the coefficients appearing in the comultiplication are positive integers, and we conjecture a combinatorial formula for these in certain cases.

The discovery of the quasisymmetric Schur functions has opened the door to many new and exciting research directions. A $q$-analogue of the Schur function basis is constructed through the use of a statistic (a quantitative measure of a certain characteristic) on semi-standard Young tableaux. We hope to determine a natural analogue of this statistic to allow us to develop a $q$-analogue of the quasisymmetric Schur functions with certain desired properties. An additional parameter, if chosen carefully could create a quasisymmetric analogue of Macdonald polynomials which would provide insight into the structure and nature of symmetric Macdonald polynomials as well as interpolate between the nonsymmetric and symmetric versions.
4 Demazure Characters and Standard Bases

The Schur functions correspond to the characters of the irreducible representations of the general linear group \([2]\). Demazure’s “Formule des caractères” \([1]\) provides an interpolation between a dominant weight corresponding to a partition \(\lambda\) and the Schur function of index \(\lambda\). For each permutation \(\mu\), one obtains a character whose description involves the Schubert variety of index \(\mu\), called the Demazure character. The “Demazure operator”

\[
D_\mu = D_{a_1} D_{a_2} \cdots D_{a_k}, \quad \text{where} \quad D_i = \frac{1 - s_i}{1 - x_i x_i^{-1}}
\]

applied to \(X^\lambda\) produces the Demazure character corresponding to \(\lambda\) and \(\mu = a_1 a_2 \cdots a_k\). Note that any given SSYT weight might appear in several different Demazure characters corresponding to \(\lambda\).

Replace \(D_i\) by the operator \(D_i - 1\) to obtain the polynomial \(\mathcal{U}(\mu, \lambda)\). This procedure produces a set of polynomials which partition those appearing in \(s_\lambda\). Considering Young tableaux as words in the free algebra, Lascoux and Schützenberger \([11]\) provide an inductive description of the standard bases \(\mathcal{U}(\mu, \lambda)\) using symmetrizing operators on the free algebra which lift the operators introduced by Demazure. The Demazure atoms provide a non-inductive combinatorial description of \(\mathcal{U}(\mu, \lambda)\) for arbitrary \(\mu, \lambda\).

**Theorem 5 (M) \([15]\)** The standard basis \(\mathcal{U}(\mu, \lambda)\) is equivalent to the Demazure atom \(A_{\mu(\lambda)}\), where \(\mu(\lambda)\) denotes the action of \(\mu\) on the parts of \(\lambda\).

The characters developed by Demazure can be considered as polynomials \(\kappa_{\mu(\lambda)}\). Each of these polynomials is the commutative image of a sum of standard bases, so these standard bases provide a decomposition of characters into “partial” characters. The representation theoretic properties of nonsymmetric Macdonald polynomials \([10]\) will be translated into this new combinatorial interpretation in a future research project. This connection permits the introduction of exciting new algebraic frontiers to describe the representations defined by the Demazure atoms. Our combinatorial description of the coefficients (in terms of objects called \(LRKs\)) in the Demazure character expansion for the product of a Demazure character and a Schur function \([8]\) provides insight into the structure of these representations.

**Theorem 6 (M) \([8]\)** Let \(\lambda\) be a partition and \(\gamma, \delta\) be weak compositions. In the expansion

\[
\kappa_\gamma \cdot s_\lambda = \sum_\delta b^{\delta \lambda}_{\gamma} \kappa_\delta,
\]

the coefficient \(b^{\delta \lambda}_{\gamma}\) is the number of LRK of shape \(\delta/\gamma\).

The Schubert polynomials are a combinatorial tool for certain questions in algebraic geometry related to Schubert varieties. They were introduced in 1982 by Lascoux and Schützenberger as the images of divided differences in a symmetric group and can be written as sums of Demazure characters \([16]\). Our multiplication rule for the product of a Schur function and a Demazure character can therefore be summed to obtain a combinatorial rule for the multiplication of a Schur function and a Schubert polynomial, which provides a new perspective from which to approach the well-known open problem of finding a combinatorial formula for the product of two Schubert polynomials.
References