

```
> restart;
```

Solve for the energies of the square wave potential. Work the problem in units where $\hbar^2/2m=a=1$.

```
> Digits:=10;
```

```
Digits := 10
```

Need the linalg package to get the "eigenvectors" function.

```
> with(linalg):
```

Warning, the protected names norm and trace have been redefined and unprotected

Define the matrix from the Central Equation derived for the square wave potential. U_{dc} is the zero frequency component of U (the average potential).

```
> M:=<<Udc+(k-2*2*Pi)^2,1/Pi,0,-1/(3*Pi),0>|  
      <1/Pi,Udc+(k-1*2*Pi)^2,1/Pi,0,-1/(3*Pi)>|  
      <0,1/Pi,Udc+k^2,1/Pi,0>|  
      <-3/Pi,0,1/Pi,Udc+(k+1*2*Pi)^2,1/Pi>|  
      <0,-3/Pi,0,1/Pi,Udc+(k+2*2*Pi)^2>>;
```

```
>
```

$$M := \begin{bmatrix} U_{dc} + (k - 4\pi)^2 & \frac{1}{\pi} & 0 & -\frac{3}{\pi} & 0 \\ \frac{1}{\pi} & U_{dc} + (k - 2\pi)^2 & \frac{1}{\pi} & 0 & -\frac{3}{\pi} \\ 0 & \frac{1}{\pi} & U_{dc} + k^2 & \frac{1}{\pi} & 0 \\ -\frac{1}{3\pi} & 0 & \frac{1}{\pi} & U_{dc} + (k + 2\pi)^2 & \frac{1}{\pi} \\ 0 & -\frac{1}{3\pi} & 0 & \frac{1}{\pi} & U_{dc} + (k + 4\pi)^2 \end{bmatrix}$$

Set U_{max} equal to 1 which means $U_{dc}=0.5$.

```
> M;
```

$$\begin{bmatrix} Udc + (k - 4\pi)^2 & \frac{1}{\pi} & 0 & -\frac{3}{\pi} & 0 \\ \frac{1}{\pi} & Udc + (k - 2\pi)^2 & \frac{1}{\pi} & 0 & -\frac{3}{\pi} \\ 0 & \frac{1}{\pi} & Udc + k^2 & \frac{1}{\pi} & 0 \\ -\frac{1}{3\pi} & 0 & \frac{1}{\pi} & Udc + (k + 2\pi)^2 & \frac{1}{\pi} \\ 0 & -\frac{1}{3\pi} & 0 & \frac{1}{\pi} & Udc + (k + 4\pi)^2 \end{bmatrix}$$

> **Udc:=0.5;**

Udc := 0.5

Pick one k value to find energies and corresponding eigenfunctions.

> **k:=0;**

k := 0

> ;

> **Eigs:=eigenvectors (M) ;**

```
Eigs := [158.4170926, 1,
  {[-1.172579950, -0.006303039368, -0.000008470145215, 0.002100982695, 0.3908948863]}],
  [39.97435866, 1,
  {[0.007493010848, -0.9564797083, -0.002789665008, 0.6105267519, -0.002497670851]}],
  [158.4136708, 1, {[0.9045544690, 3.06320 10-7, 2.91743 10-10, -1.021669 10-7, 0.3014801449]}],
  [0.49486776, 1, {[0.00004595986430, 0.01139916216, -1.414057972, 0.01140015056, -0.0000153173]}],
  [39.98418682, 1,
  {[-0.002013620622, -0.8189654580, -0.01081623943, -0.5227158349, 0.000671206464]}]
```

The above is a list of five entries for the five energy eigenvalues and eigenvectors.

The first entry in each list item is then energy eigenvalue.

The next entry is the degeneracy. All our eigenvectors are non-degenerate.

The next entry is the eigenvector, telling us how much (C_i) of the each of the five $\psi(k+G)$ basis functions.

We need to build a sum of $C_i * \psi_i$ over the five wave functions.

Here, recall that adding a subscript "[3]" to a list pulls out the third item in the list. For example, to

get the fourth component of the second eigenvector, we would have a sequence

[2] -- second eigenvector

[3] -- the third entry in this eigenvector entry, which is the set of one eigenvector in the entry.

[1] -- selects the first and only eigenvector in this list. (This just strips off the curly brackets from the eigenvector.)

[4] -- selects the fourth component of this vector.

For example:

```
> psi:=Eigs[2][3][1][4]*exp(I*(k-2*Pi)*x);  
psi:=0.6105267519 e(-2 I π x~)
```

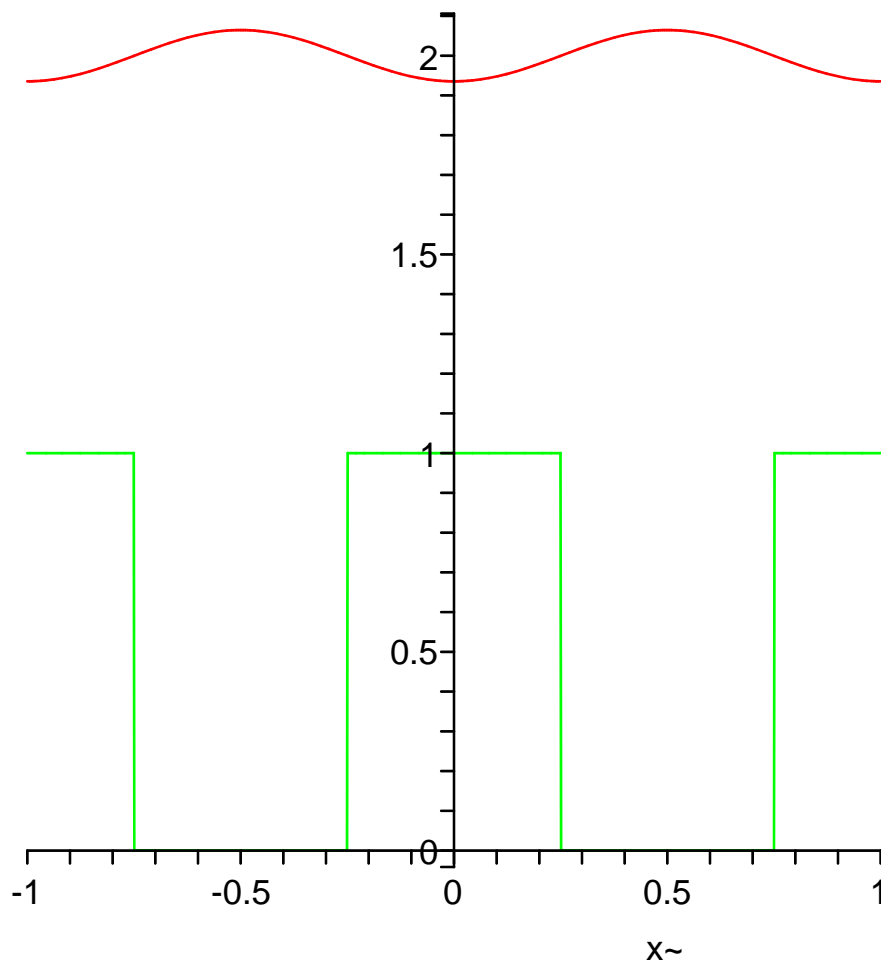
Now construct our $G=2\pi j/a$ sum, $h=-2..2$. Note that $j=-2$ is eigenvector 1, so I need to add 3 to convert j to a subscript.

Let's do it first for the ground state, the fourth eigenvalue/eigenvector.

```
> psi2:=sum(Eigs[4][3][1][j+3]*exp(I*(k-j*2*Pi)*x),j=-2..2);  
psi2 := -1.414057972 + 0.00004595986430 e(4 I π x~) + 0.01139916216 e(2 I π x~) + 0.01140015056 e(-2 I π x~)  
- 0.0000153173 e(-4 I π x~)
```

Now plot the result, along with the potential. Note that Heaviside(cos) gives me the desired square potential.

```
> plot({Heaviside(cos(2*Pi*x)),conjugate(psi2)*psi2},x=-1..1);
```



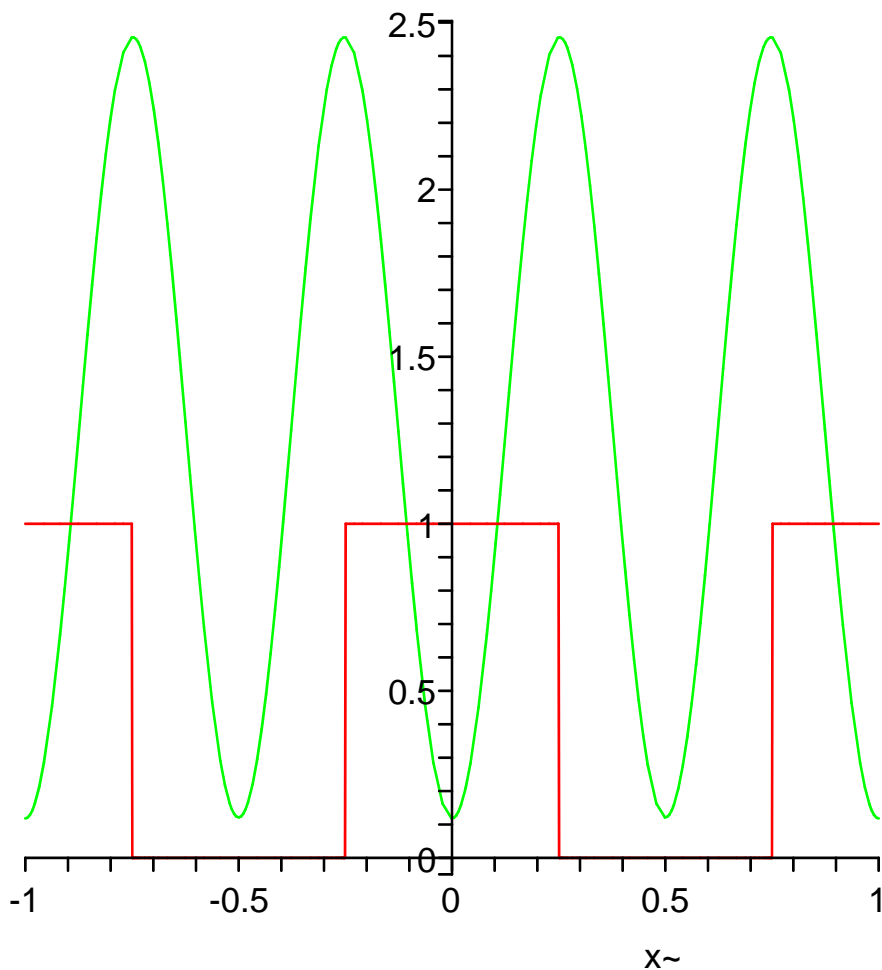
We can see why it is a low energy state. The charge is piled up in the potential wells. (Ignore the average height. The wave function is not normalized.)

Now let's look at next lowest state, the second eigenvector.

```
> psi2:=sum(Eigs[2][3][1][j+3]*exp(I*(k-j*2*Pi)*x),j=-2..2);  
psi2 := -0.002789665008 + 0.007493010848 e(4 I π x~) - 0.9564797083 e(2 I π x~) + 0.6105267519 e(-2 I π x~)  
- 0.002497670851 e(-4 I π x~)
```

Now plot the result, along with the potential. Note that Heaviside(cos) gives me the desired square potential.

```
> plot({Heaviside(cos(2*Pi*x)),conjugate(psi2)*psi2},x=-1..1);
```



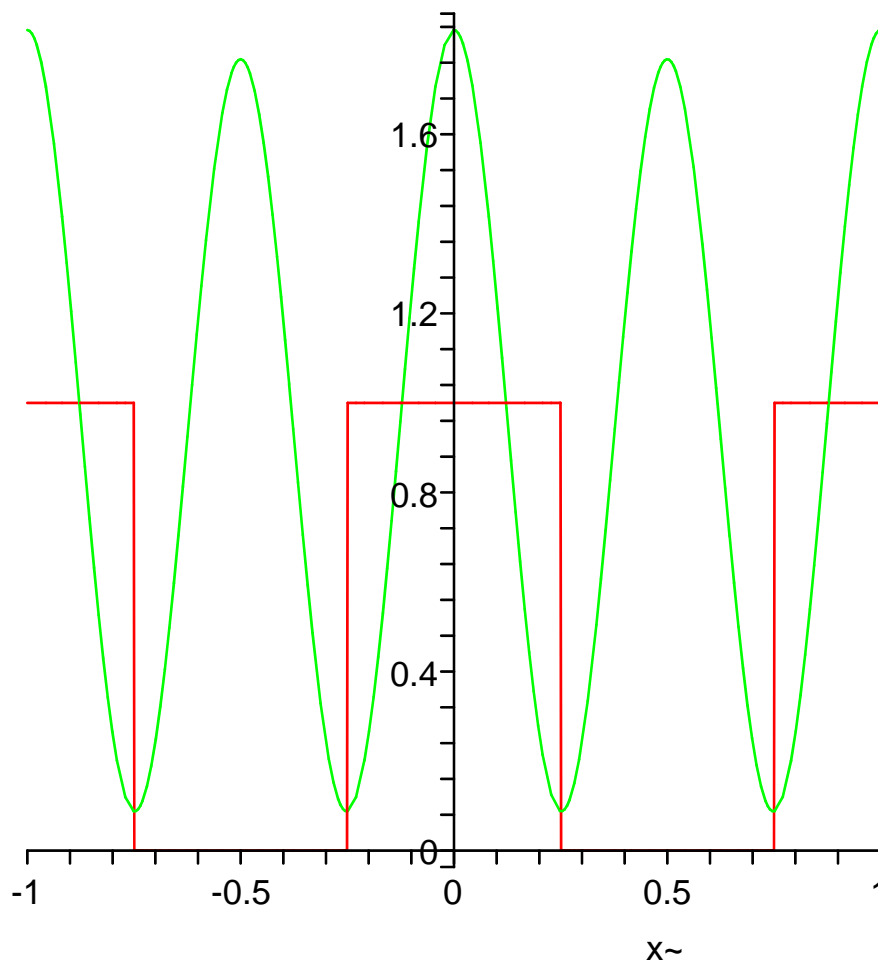
>

Let's look at the next, eigenvector 5:

```
> psi2:=sum(Eigs[5][3][1][j+3]*exp(I*(k-j*2*Pi)*x),j=-2..2);  
psi2 := -0.01081623943 - 0.002013620622 e(4 I π x~) - 0.8189654580 e(2 I π x~) - 0.5227158349 e(-2 I π x~)  
+ 0.000671206464 e(-4 I π x~)
```

Now plot the result, along with the potential. Note that Heaviside(cos) gives me the desired square potential.

```
> plot({Heaviside(cos(2*Pi*x)), conjugate(psi2)*psi2}, x=-1..1);
```



We can see why 2 and 5 are higher in energy than the ground state, and why they are about the same energy. Both distribute the charge nearly equally in the barriers and in the wells, so their potentials are about the same and are higher in frequency and so higher in kinetic energy.

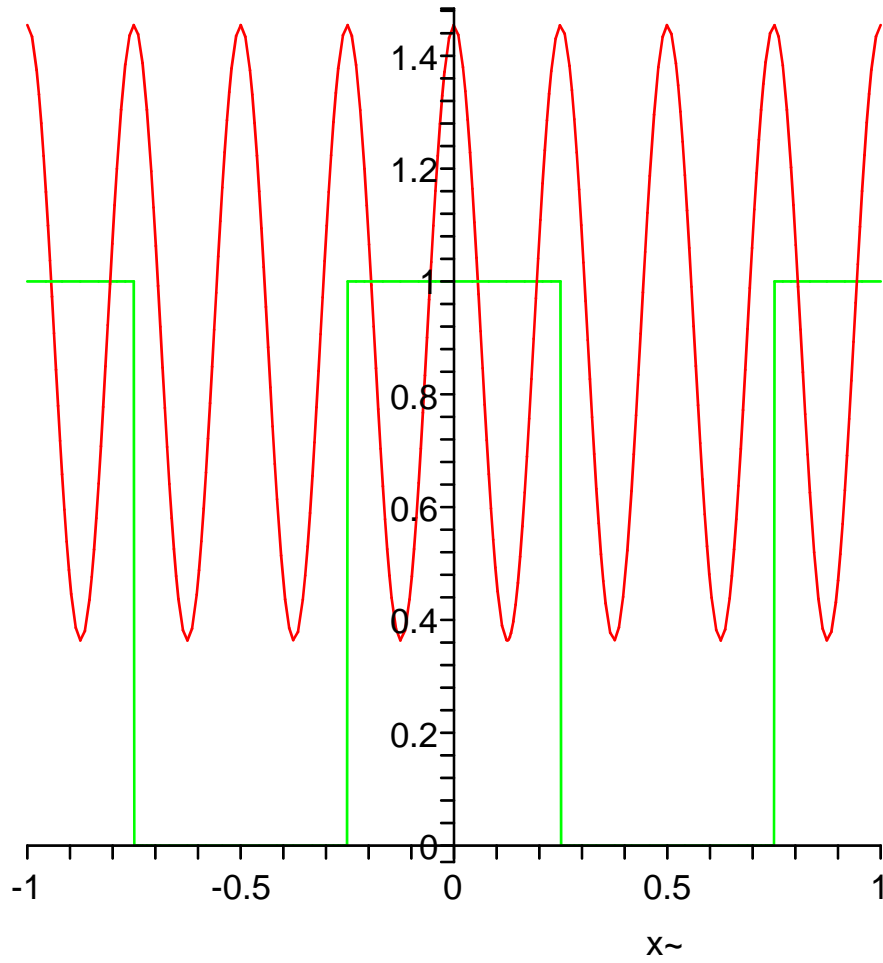
For fun, let's look at the next, eigenvector 3.

```
> psi2:=sum(Eigs[3][3][1][j+3]*exp(I*(k-j*2*Pi)*x), j=-2..2);
```

$$\psi_2 := 2.91743 \cdot 10^{-10} + 0.9045544690 e^{(4 I \pi x)} + 3.06320 \cdot 10^{-7} e^{(2 I \pi x)} - 1.021669 \cdot 10^{-7} e^{(-2 I \pi x)} + 0.3014801449 e^{(-4 I \pi x)}$$

Now plot the result, along with the potential. Note that Heaviside(cos) gives me the desired square potential.

```
> plot({Heaviside(cos(2*Pi*x)), conjugate(psi2)*psi2}, x=-1..1);
```



The potential energy of this charge density is similar to that of 2 and 5, but the frequency is higher and so therefore is the kinetic energy.