Problem 1.3

a.

One way to address the problem is to unwrap the cylinder into a flat rectangle of width 1 and infinite height. For simplicity of the discussion, let's orient this infinite rectangle vertically.

What constraints does this put on the lattice vectors? The obvious case is to pick perpendicular lattice vectors, parallel to and perpendicular to the sides of the rectangle. Then we just need for $\mathbf{a_1}$ to be horizontal and $\mathbf{a_2}$ to be vertical. The only constraint is that the length a_1 must be a submultiple of 1, $a_1 = 1/n$, for any integer *n*. Then a_2 can be anything.

But this is not the only possible lattice. Our minimum condition is

$$m_1\mathbf{a_1} + m_2\mathbf{a_2} + (1,0) = m_3\mathbf{a_1} + m_4\mathbf{a_2}.$$

for any integers $m_1,...,m_4$. So

$$m_1 a_{1x} + m_2 a_{2x} + 1 = m_3 a_{1x} + m_4 a_{2x}.$$

$$m_1 a_{1y} + m_2 a_{2y} + 0 = m_3 a_{1y} + m_4 a_{2y}.$$

Define $n_1 \equiv m_3 - m_1,$

$n_2 \equiv m_4 - m_2.$

Then

$$n_1 a_{1x} + n_2 a_{2x} = 1.$$

$$n_1 a_{1y} + n_2 a_{2y} = 0.$$

These two arbitrary integers n_1 and n_2 , along with the horizontal period, place constraints on the lattice vectors. The simple obvious case follows from picking one of these two integers to be zero. Then the one lattice vector is in the x direction and the other is in the y direction.

But we can also have both non-zero. This gives us non-orthogonal lattice vectors and a helical lattice, as are found in nanotubes.

b.

For a torus, we now have a finite height rectangle with periodic boundary conditions in both x and y. Similarly to part a, we have

$$m_1 a_{1x} + m_2 a_{2x} + 1 = m_3 a_{1x} + m_4 a_{2x},$$
$$m_1 a_{1y} + m_2 a_{2y} + 0 = m_3 a_{1y} + m_4 a_{2y},$$

and

$$m_5 a_{1x} + m_6 a_{2x} + 1 = m_7 a_{1x} + m_8 a_{2x},$$
$$m_5 a_{1y} + m_6 a_{2y} + 0 = m_7 a_{1y} + m_8 a_{2y}.$$

Define $n_1 \equiv m_3 - m_1$,

$$n_2 \equiv m_4 - m_2,$$
$$n_3 \equiv m_7 - m_5,$$
$$n_4 \equiv m_8 - m_6.$$

Then

$$n_1 a_{1x} + n_2 a_{2x} = 1.$$
$$n_1 a_{1y} + n_2 a_{2y} = 0.$$

and

$$n_3 a_{1x} + n_4 a_{2x} = 0.$$
$$n_3 a_{1y} + n_4 a_{2y} = 1.$$

Pick any set of these four n_i's and you can solve for the components of the lattice vectors.

Problem 1.4.

Choose one lattice vector to be $\mathbf{T_1} = (a, 0)$ on the
x axis. If we have a symmetry rotation of θ, then
a second primitive lattice vector is $\mathbf{T_2} = (a\cos\theta, a\sin\theta)$.

Rotating $\mathbf{T_1}$ clockwise by θ gives the point $\mathbf{T_3} = (a\cos\theta, -a\sin\theta)$.
Then $\mathbf{T_4} = \mathbf{T_2} - \mathbf{T_3} = (0, 2a\sin\theta)$ must also be in the Bravais lattice.
It can only be in the Bravais lattice if there exist two integers
m_1 and m_2 such that
$m_1\mathbf{T_1} + m_2\mathbf{T_2} = \mathbf{T_4}$, or
$m_1(a,0) + m_2(\cos\theta, \sin\theta) = (0, 2a\sin\theta)$
$\Rightarrow m_2 = 2$
$\Rightarrow 0 = m_1 + 2\cos\theta$
$\Rightarrow \cos\theta = \frac{m_1}{2}$. In other words, $\cos\theta$ must be a half
integer. The only possibilities are
$0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$, and π (and of course their negatives).
These correspond to the identity operation, a six-fold
axis, a four-fold axis, a three-fold axis, and a two-fold
axis. No other possibilities.
This proves parts a and b.

1.5 a. See Maple output.

Symmetry problem:

Square:

Identity operation
Rotations of 90, 180, 270 degrees
Horizontal, vertical, and two diagonal mirror planes.
Inversion symmetry.

Hexagonal:

Identity
Rotations of 60, 90, 120, 180, 240, 270, 300 degrees.
Six mirror planes.

Inversion symmetry.

Rectangular:

Identity
Rotation of 180 degrees.
Horizontal and vertical mirror planes.
Inversion.

Centered Rectangular:

Identity
Rotation of 180 degrees.
Horizontal and vertical mirror planes.
Inversion.

Oblique:

Identity
Rotation of 180 degrees.
Inversion