PHY 711 – Lecture notes on Lagrangian for Electric and Magnetic Fields

For simplicity, consider a Lagrangian for a single particle having the form (in Cartesian coordinates)
\[ L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U. \]
The Euler-Lagrange equations have the form:
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad (1) \]
with similar equations for \( y \) and \( z \). We can show that this form is consistent with Newton’s Laws if the potential function \( U \) takes the form:
\[ U = U^0(x, y, z, t) + U^{EM}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \quad (2) \]
where \( U^{EM} \) represents the interaction of our particle (having charge \( q \)) with an electric field \( E \) and magnetic field \( B \) where we can represent the fields in terms of the scalar and vector potentials:
\[ E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t} \quad \text{and} \quad B = \nabla \times A. \quad (3) \]
We must find \( U^{EM} \) which is both consistent with the Euler-Lagrange Eq.(1) and with the Lorentz force (written in the \( x \) direction):
\[ F_x = q(E_x + \frac{1}{c} (\dot{\mathbf{r}} \times \mathbf{B})|_x) = -\frac{\partial U^{EM}}{\partial x} + \frac{d}{dt} \left( \frac{\partial U^{EM}}{\partial \dot{x}} \right). \quad (4) \]
We note that the magnetic field terms can be evaluated:
\[ \dot{\mathbf{r}} \times (\nabla \times A)|_x = \dot{y} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \dot{z} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right). \quad (5) \]
The right hand side of Eq.(5) (with the addition and subtraction of a convenient term) can be written:
\[ \dot{\mathbf{r}} \cdot (\nabla \times A)|_x = \dot{y} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \dot{z} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right), \quad (6) \]
where we are assuming that \( A_x = A_x(x, y, z, t) \). Noting that \( A_x = \partial(\dot{\mathbf{r}} \cdot \mathbf{A})/\partial \dot{x} \), the electromagnetic force can thus be written:
\[ F_x = -q \frac{\partial \phi}{\partial x} - \frac{q}{c} \frac{\partial A_x}{\partial t} + q \frac{\partial}{\partial t} \left( \frac{\partial (\dot{\mathbf{r}} \cdot \mathbf{A})}{\partial x} - \frac{d}{dt} \frac{\partial (\dot{\mathbf{r}} \cdot \mathbf{A})}{\partial \dot{x}} + \frac{\partial A_x}{\partial t} \right). \quad (7) \]
Simplifying this equation, we obtain
\[ F_x = -\frac{\partial}{\partial x} \left( q\phi - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A} \right) - \frac{d}{dt} \frac{\partial}{\partial \dot{x}} \left( \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A} \right). \quad (8) \]
Thus, we finally have the result
\[ U^{EM} = q\phi - \frac{q}{c} \dot{\mathbf{r}} \cdot \mathbf{A}. \quad (9) \]