

PHY 113 – Additional notes for Chapter 21 (Lectures 28 & 29)

Expressions for the internal energy of an ideal gas

In class, we did not quite have enough time to discuss the form of the internal energy and the relations of adiabatic processes for ideal gases. Hopefully some of the following comments will be helpful.

In class in Eq. 21.2 of your text, we showed that the pressure P that N ideal gas molecules exerts on the walls of a container of volume V is related to the average squared velocity $\langle v^2 \rangle$ of the gas molecules having mass m according to:

$$PV = \frac{1}{3}Nm\langle v^2 \rangle = \frac{2}{3}N\langle K_{\text{trans}} \rangle, \quad (1)$$

where the translational kinetic energy of the molecule is given by: $\langle K_{\text{trans}} \rangle = \frac{1}{2}m\langle v^2 \rangle$. From the ideal gas law, we also know that $PV = Nk_B T$, which allows to relate the average translational kinetic energy of a molecule to its temperature:

$$\frac{2}{3}N\langle K_{\text{trans}} \rangle = Nk_B T, \quad (2)$$

or

$$\langle K_{\text{trans}} \rangle = \frac{3}{2}k_B T. \quad (3)$$

This result, by itself is very interesting, since it relates the average kinetic energy and the average squared velocity of a gas molecule to the temperature. Although the derivation, was very approximate, more detailed derivations are consistent with this result.

An important extension of this result is the relationship of temperature to the internal energy of the ideal gas. In addition to the translational energy, more complicated motions of molecules such as rotations contribute to this internal energy at room temperature. (In general, it turns out that molecular vibrations are not important at room temperature.) There is a notion of “degrees of freedom” that relates to the molecular shape. The degrees of freedom f is 3, 5, and 6 for an atom, a linear molecule, and a non-linear triatomic molecule, respectively. A very rough argument allows us to say that each degree of freedom is associated with a contribution of $\frac{1}{2}k_B T$ to the internal energy so that

$$E_{\text{int}} = \frac{f}{2}Nk_B T. \quad (4)$$

Operationally, it turns out that it is convenient to calculate the internal energy in terms of the γ factor which is determined experimentally for each ideal gas:

$$\gamma = \frac{C_P}{C_V}. \quad (5)$$

In terms of γ , the internal energy is given by:

$$E_{\text{int}} = \frac{1}{\gamma - 1} N k_B T. \quad (6)$$

The proof of Eq. (6) goes as follows. Let us assume that Eq. (4) is true. Suppose we heat a gas at constant volume. No work is done in this process so that $\Delta E_{\text{int}} = \Delta Q$. We can then solve for the heat change of this process and relate it to the heat capacity at constant volume c_v per molecule:

$$\Delta Q]_{\text{constant V}} = \frac{f}{2} N k_B \Delta T \equiv N c_v \Delta T. \quad (7)$$

From this relation, we see that $c_v = \frac{f}{2} k_B$. Now suppose we heat the same gas at constant pressure. In this case, work in the amount of $P \Delta V = N k_B \Delta T$ is done by the system. So for this case, the first law of thermodynamics tells us that

$$\Delta Q]_{\text{constant P}} = \frac{f}{2} N k_B \Delta T + N k_B \Delta T \equiv N c_p \Delta T. \quad (8)$$

From this relation, we see that $c_p = \frac{f}{2} k_B + k_B = \frac{f+2}{2} k_B$. From these two results, we then have that $\gamma \equiv \frac{c_p}{c_v}$ is related to the degrees of freedom f according to:

$$\frac{f}{2} = \frac{1}{\gamma - 1}. \quad (9)$$

From this result, we then see that Eq. (6) then follows and that we can then use expressions 6 and 4 interchangeably to evaluate the internal energy. Eq. (6) is often convenient, since the γ parameter can often be determined from experiment.

Expressions for adiabatic processes of an ideal gas

For an adiabatic process, $\Delta Q = 0$, so that according to the first law of thermodynamics, $\Delta E_{\text{int}} = -\Delta W$. In order to analyze how this affects the change of temperature, volume, and pressure, we can write down the following expressions.

$$\frac{1}{\gamma - 1} N k_B \Delta T = -P \Delta V. \quad (10)$$

Since we also know that $PV = N k_B T$, we can eliminate one of the variables P , V , or T from Eq. (10), and derive a relation between the other two. For example, if we replace T by $T = \frac{PV}{N k_B}$, then equation 10 reads

$$\frac{1}{\gamma - 1} \Delta(PV) = -P \Delta V. \quad (11)$$

Here, we note that $\Delta(PV) = \Delta PV + P\Delta V$. Putting all of the same types of variables the same side of the equation and doing a little bit of algebra, we find that

$$\frac{\Delta P}{P} = -\gamma \frac{\Delta V}{V}. \quad (12)$$

Integrating this differential equation, we find that

$$\ln P = -\gamma \ln V + \text{constant}, \quad (13)$$

which can be written as:

$$PV^\gamma = \text{constant}. \quad (14)$$

Since $PV = Nk_B T$, we can express the adiabatic equation of state in terms of the other variables as

$$TV^{\gamma-1} = \text{constant} \quad (15)$$

or

$$P^{(1/\gamma-1)}T = \text{constant}. \quad (16)$$