September 16, 2004

PHY 742 – Notes on Clebsch-Gordan coefficients

Reference: Abramowitz & Stegum – pg. 1006

Original formula:

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = \delta_{m,m_1+m_2} \sqrt{\frac{(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!(2j+1)}{(j_1+j_2+j+1)!}} \tag{1}$$
$$\times \sum_k \frac{(-1)^k \sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!}}{k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j-j_2+m_1+k)!(j-j_1-m_2+k)!}$$

Working formula:

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = \delta_{m,m_1+m_2} \sqrt{\frac{(j_1-j_2+j)!(-j_1+j_2+j)!(2j+1)}{(j_1+j_2-j)!(j_1+j_2+j+1)!}} \tag{2}$$
$$\times \sqrt{\frac{(j_1+m1)!(j_2-m_2)!(j+m)!}{(j_1-m1)!(j_2+m_2)!(j-m)!}}$$
$$\times \sum_k \frac{(-1)^k}{k!} \frac{(j_1+j_2-j)!}{(j_1+j_2-j-k)!} \frac{(j_1-m_1)!}{(j_1-m_1-k)!} \frac{(j_2+m_2)!}{(j_2+m_2-k)!} \frac{(j-m)!}{(j-j_2+m_1+k)!(j-j_1-m_2+k)!}$$

Additional reference for Clebsch-Gordan coefficients: Complement $\mathbf{B}_X$ and $\mathbf{C}_X$ of Cohen-Tannoudji's text (Volume #2).

A related quantity is the Gaunt coefficient:

$$G^{LM}_{l_1 m_1 l_2 m_2} \equiv \sqrt{4\pi} \int d\Omega Y^*_{l_1 m_1}(\hat{\mathbf{r}}) Y^*_{LM}(\hat{\mathbf{r}}) Y_{l_2 m_2}(\hat{\mathbf{r}}).$$

As discussed in Cohen-Tannoudji's text, the Gaunt coefficent can be expressed in terms of a product of two Clebsch-Gordan coefficients:

$$G^{LM}_{l_1 m_1 l_2 m_2} = (-1)^{m_2} \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2L+1)}} \langle l_1\ l_2\ 0\ 0 | L\ 0\rangle \langle l_1\ l_2\ m_1\ -m_2 | L\ M\rangle. \tag{3}$$