PHY 113 A General Physics I
9-9:50 AM MWF Olin 101

Plan for Lecture 21:

Chapter 15 – Simple harmonic motion

1. **Object attached to a spring**
   - **Displacement as a function of time**
   - **Kinetic and potential energy**
2. **Pendulum motion**

14	10/03/2012	Momentum and collisions	9.5-9.9	9.29,9.37	10/05/2012
	10/05/2012	Review	6-9		
	10/08/2012	Exam	6-9		
15	10/10/2012	Rotational motion	10.1-10.5	10.6, 10.13, 10.25	10/12/2012
16	10/12/2012	Torque	10.6-10.9	10.37, 10.55	10/15/2012
17	10/15/2012	Angular momentum	11.1-11.5	11.11, 11.34	10/17/2012
18	10/17/2012	Equilibrium	12.1-12.4	12.11, 12.39	10/22/2012
	10/19/2012	*Fall Break*			
19	10/22/2012	Simple harmonic motion	15.1-15.3	15.4, 15.20	10/24/2012
20	10/24/2012	Resonance	15.4-15.7	15.43, 15.43, 15.52	10/26/2012
21	10/26/2012	Gravitational force	13.1-13.3	13.6, 13.10, 13.13	10/29/2012
22	10/29/2012	Kepler's laws and satellite motion	13.4-13.6	13.28, 13.34	10/31/2012
	10/31/2012	Review	10-13,15		
	11/02/2012	Exam	10-13,15		
23	11/05/2012	Fluid mechanics	14.1-14.4		11/07/2012
24	11/07/2012	Fluid mechanics	14.5-14.7		11/09/2012
25	11/09/2012	Temperature	19.1-19.5		11/12/2012

iclicker exercise:

Why did your textbook develop a whole chapter on oscillatory motion?

A. **Because the authors like to torture students.**
B. **Because it is different from Newton's laws and needs many pages to explain.**
C. **Because it is an example of Newton's laws and needs many pages to explain.**

iclicker exercise:

Simple harmonic motion has a characteristic time period T. What determines T?

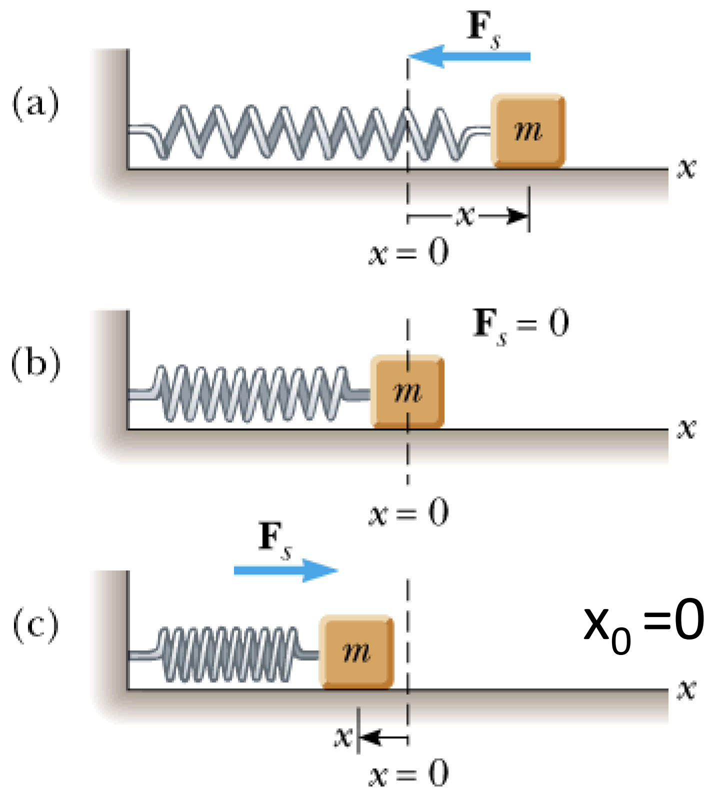
A. **The characteristics of the physical system.**
B. **The initial displacements or velocities.**
C. **It is not possible to know T.**

Behavior of materials:

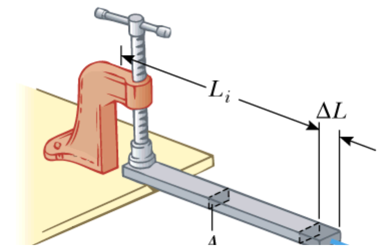
Hooke's law

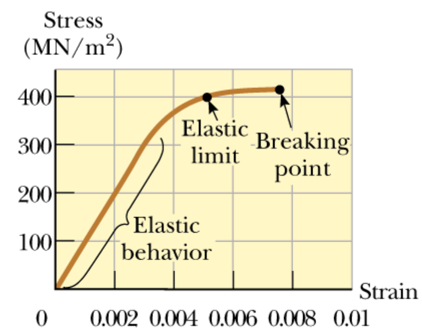
$$F_s = -k(x-x_0)$$

Serway, Physics for Scientists and Engineers, 5/e
Figure 13.1



$x_0 = 0$





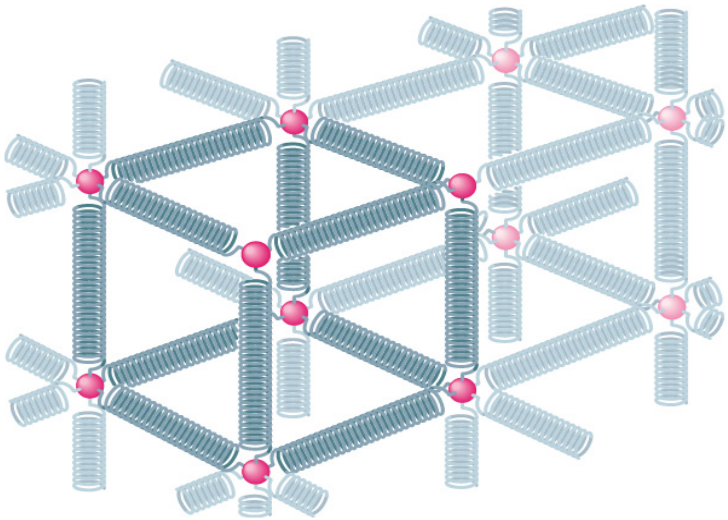
Young's modulus

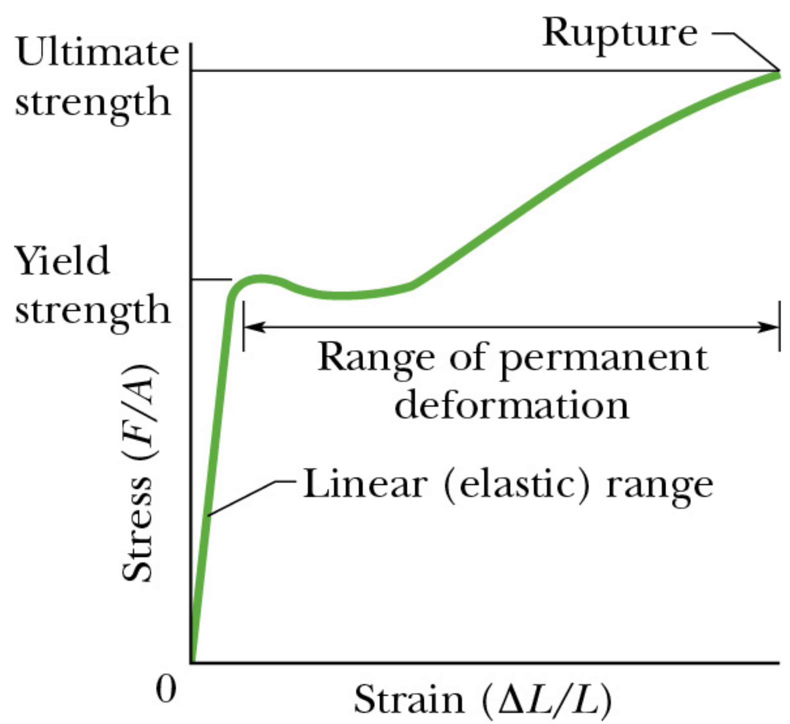
$$E = \frac{F_{applied} / A}{\Delta L / L}$$

$$F_{material} = -F_{applied}$$

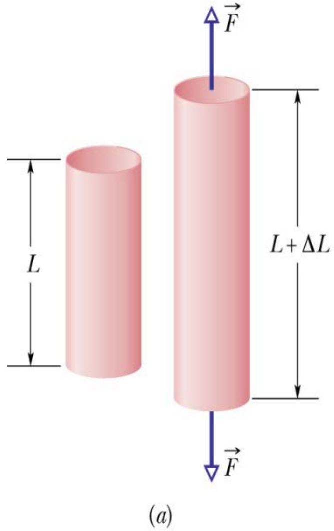
$$\Rightarrow F_{material} = -\left(\frac{EA}{L}\right)\Delta L$$

Microscopic picture of material with springs representing bonds between atoms





Measurement of elastic response:

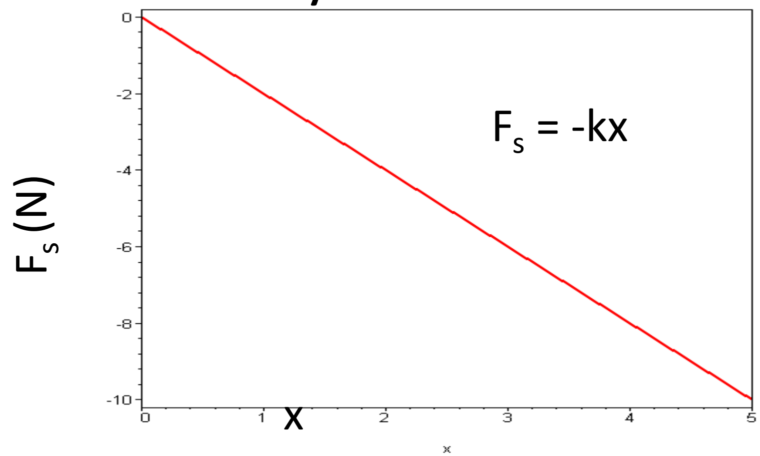


$$E = \frac{F_{applied} / A}{\Delta L / L}$$

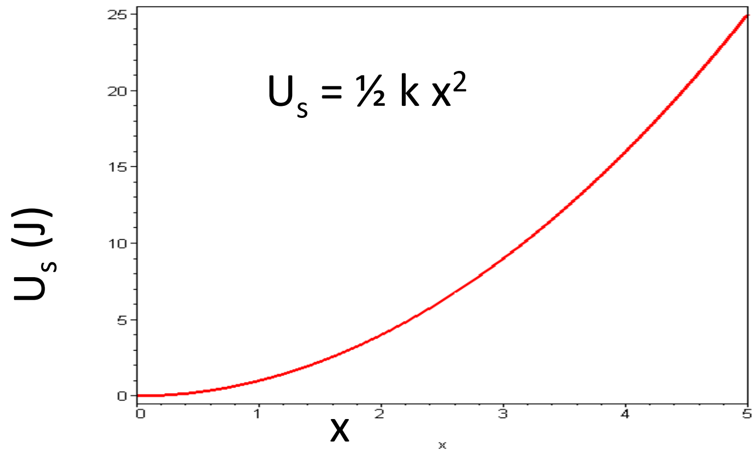
$$F_{material} = -F_{applied}$$

$$\Rightarrow F_{material} = -\overbrace{\left(\frac{EA}{L}\right)}^{k} \Delta L$$

Form of Hooke's law for ideal system:

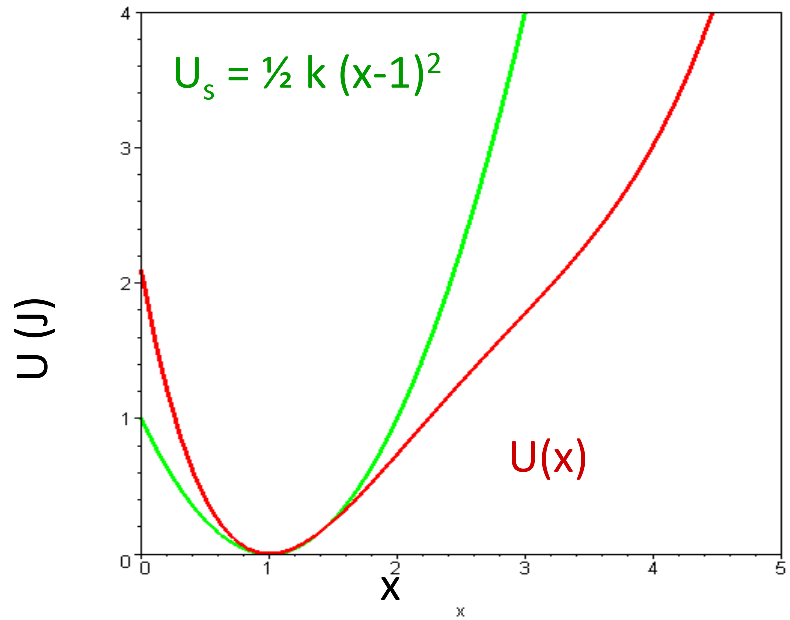


Potential energy associated with Hooke's law:



General potential energy curve:

$$k = \frac{d^2U}{dx^2}(x=1)$$



In addition to the spring-mass system, Hooke’s law approximates many physical systems near equilibrium.

Motion associated with Hooke's law forces

Newton's second law:

F = -k x = m a

$$F = -kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$ ➔ "second-order" linear differential equation

How to solve a second order linear differential equation:

Earlier example – constant force F_0 ➔ acceleration a_0

$$\frac{d^2x}{dt^2} = \frac{F_0}{m} \equiv a_0$$

$x(t) = x_0 + v_0 t + ½\, a_0\, t^2$

2 constants (initial values)

Hooke's law motion:

$$F = -kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Forms of solution:

$$x(t) = A\cos(\omega t + \varphi)$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

where: $\omega \equiv \sqrt{\frac{k}{m}}$

2 constants (initial values)

Verification:

Differential relations:

$$\frac{d\sin(\omega t+\varphi)}{dt}=\omega\cos(\omega t+\varphi)$$

$$\frac{d\cos(\omega t+\varphi)}{dt}=-\omega\sin(\omega t+\varphi)$$

Therefore:

$$\frac{d^2 A\cos(\omega t+\varphi)}{dt^2}=-\omega^2\ A\cos(\omega t+\varphi)$$

$$\Rightarrow x(t)=A\cos(\omega t+\varphi) \qquad \textit{satisfies}$$

$$\frac{d^2x}{dt^2}=-\frac{k}{m}x \qquad \textit{provided} \qquad \textit{that} \qquad \omega^2=\frac{k}{m}$$

Recap: Newton's law for mass-spring system:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Guess that solution for $x(t)$ has the form:

$x(t) = A\cos(\omega t + \varphi)$ where A and φ and ω are unknown constants

Condition that guess satisfies the equation:

$$\frac{d^2\left[A\cos(\omega t + \varphi)\right]}{dt^2} = -\omega^2\left[A\cos(\omega t + \varphi)\right] = -\frac{k}{m}\left[A\cos(\omega t + \varphi)\right]$$

$$\Rightarrow \omega^2 = \frac{k}{m}$$ (determines ω)

iclicker exercise:

Which of the following other possible guesses would provide a solution to Newton's law for the mass-spring system:

A. $A\sin(\omega t+\varphi)$

B. $A\cos(\omega t)+B\sin(\omega t)$

C. $v_0 t+\frac{1}{2}gt^2$

D. $\exp(\omega t+\varphi)$

E. More than one of the above.

$$\text{Note that: } A\sin(\omega t+\varphi)=A\cos(\varphi)\sin(\omega t)+A\sin(\varphi)\cos(\omega t)$$

$$A\cos(\omega t+\varphi)=A\cos(\varphi)\cos(\omega t)-A\sin(\varphi)\sin(\omega t)$$

Complete solution for Newton's law for mass-spring system:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

We now know that the solution for $x(t)$ has the form:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right)$$ where A and φ are unknown constants

Finding A and φ from initial conditions:

Suppose we know that

$$x(t=0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(t=0) = 0$$

$$x(t=0) = x_0 = A\cos(\varphi) \qquad \frac{dx}{dt}(t=0) = 0 = -A\sin(\varphi)$$

$$\Rightarrow \varphi = 0 \quad \text{and} \quad A = x_0 \qquad \Rightarrow x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Complete solution for Newton's law for mass-spring system:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

We now know that the solution for $x(t)$ has the form:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right) \quad \text{where } A \text{ and } \varphi \text{ are unknown constants}$$

Finding A and φ from initial conditions:

Suppose we know that

$$x(t=0) = 0 \quad \text{and} \quad \frac{dx}{dt}(t=0) = v_0$$

$$x(t=0) = 0 = A\cos(\varphi) \qquad \frac{dx}{dt}(t=0) = v_0 = -A\sqrt{\frac{k}{m}}\sin(\varphi)$$

$$\Rightarrow \varphi = \frac{\pi}{2} \quad \text{and} \quad A = -\frac{v_0}{\sqrt{\frac{k}{m}}} \qquad \Rightarrow x(t) = \frac{v_0}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

“Simple harmonic motion” in practice

A block with a mass of 0.2 kg is connected to a light spring for which the force constant is 5 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 0.05 m from equilibrium and released from rest. Find its subsequent motion.

$$\omega = \sqrt{k/m} = \sqrt{5/0.2} \text{ rad/s} = 5 \text{ rad/s}$$

x(t)= A cos (ωt+ϕ) x(0) = A cos (ϕ) = 0.05 m

v(t)=-Aωsin (ωt+ϕ) v(0)=-Aωsin (ϕ) = 0 m/s

➔ $\phi = 0$ and A = 0.05 m

iclicker exercise:

A certain mass m on a spring oscillates with a characteristic frequency of 2 cycles per second. Which of the following changes to the mass would increase the frequency to 4 cycles per second?

(a) 2m (b) 4m (c) m/2 (d) m/4

Summary --

Simple harmonic motion:

$$F = -kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Conveniently evaluated in *radians*

$$x(t) = A\cos(\omega t + \varphi); \quad \omega = \sqrt{\frac{k}{m}}$$

Constants

Note that:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$$

$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \varphi)$$

Energy associated with simple harmonic motion

Form of displacement:

$$x(t) = A\cos(\omega t + \varphi) \quad \text{where } \omega = \sqrt{\frac{k}{m}} \quad \text{and } A \text{ and } \varphi \text{ are constants}$$

Energy:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$v(t) = \frac{dx}{dt} = -A\omega \sin(\omega t + \varphi)$$

$$\Rightarrow E = \frac{1}{2}m\omega^2 A^2\left(\sin(\omega t + \varphi)\right)^2 + \frac{1}{2}kA^2\left(\cos(\omega t + \varphi)\right)^2$$

Energy associated with simple harmonic motion

Energy:

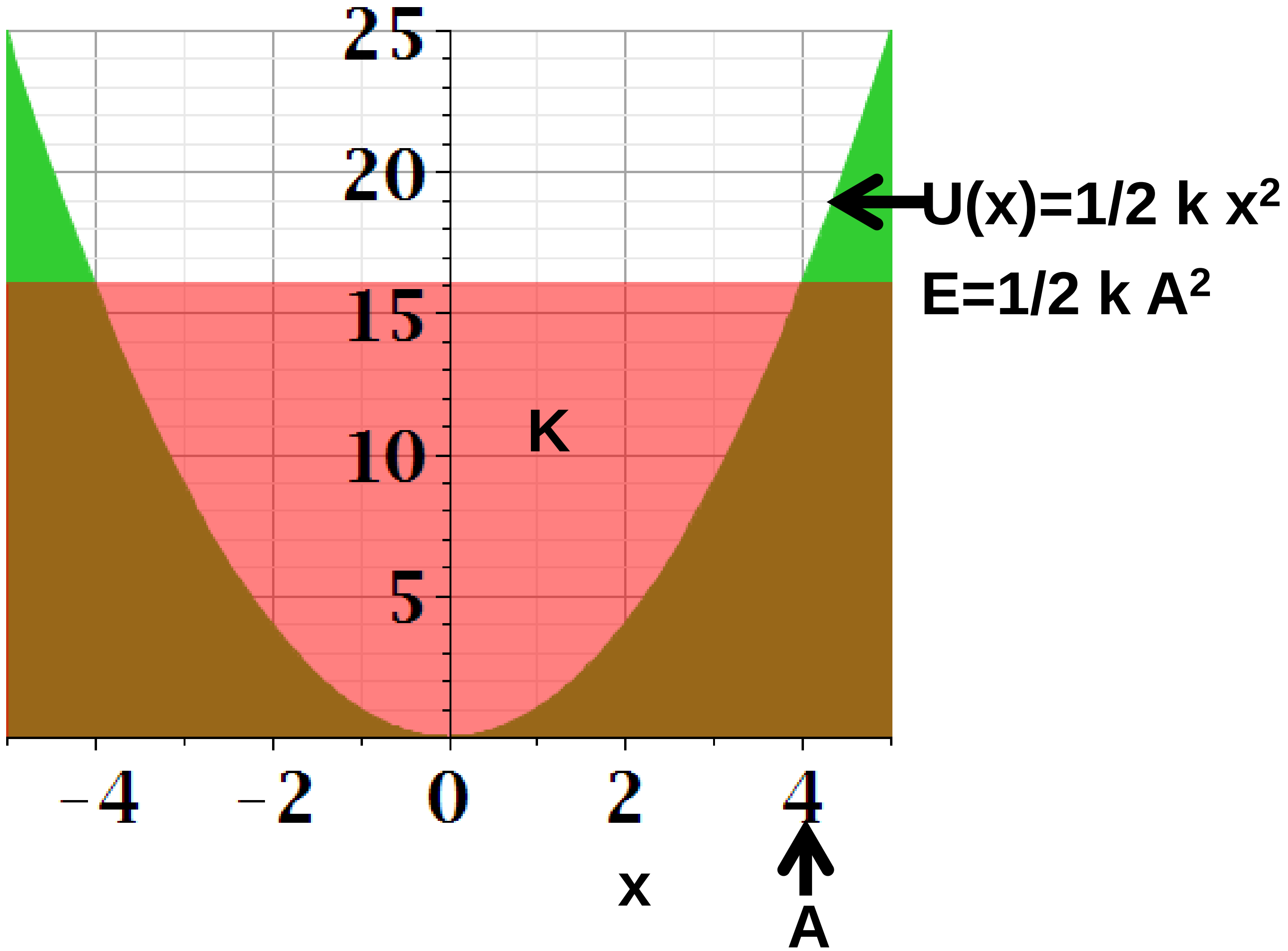
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$E = \frac{1}{2}m\omega^2 A^2 \left(\sin(\omega t + \varphi)\right)^2 + \frac{1}{2}kA^2 \left(\cos(\omega t + \varphi)\right)^2$$

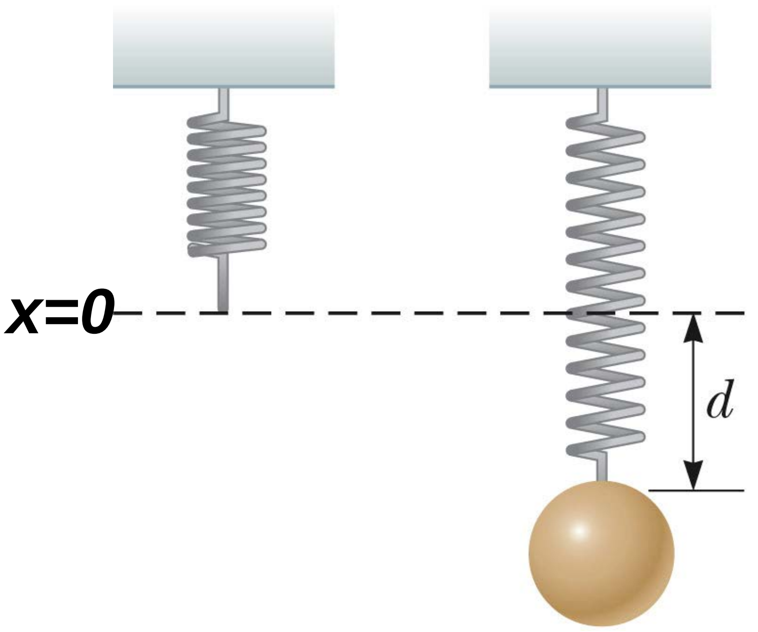
But $\omega^2 = \frac{k}{m}$

$$\Rightarrow E = \frac{1}{2}kA^2 \left[\left(\sin(\omega t + \varphi)\right)^2 + \left(\cos(\omega t + \varphi)\right)^2\right] = \frac{1}{2}kA^2$$

Energy diagram:



Effects of gravity on spring motion



$$\text{At equilibrium:} \quad kd - mg = 0$$

$$d = \frac{mg}{k}$$

$$F = -kx - mg = m\frac{d^2x}{dt^2}$$

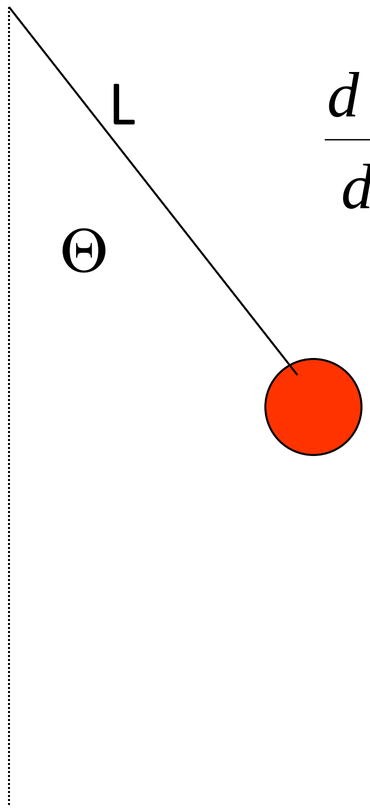
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - g$$

Rewriting :

$$\frac{d^2(x+d)}{dt^2} = -\frac{k}{m}\left(x + \frac{mg}{k}\right) = -\frac{k}{m}(x+d)$$

$$\Rightarrow x(t) = -d + A\cos\left(\sqrt{\frac{k}{m}}t + \varphi\right)$$

Simple harmonic motion for a pendulum:



$$\tau = mgL\sin\Theta = -I\alpha = -I\,\frac{d^2\Theta}{dt^2}$$

$$\frac{d^2\Theta}{dt^2} = -\frac{mgL}{I}\sin\Theta = -\frac{g}{L}\sin\Theta \quad (\text{since } I = mL^2)$$

Approximation for small Θ:

$$\sin\Theta \approx \Theta$$

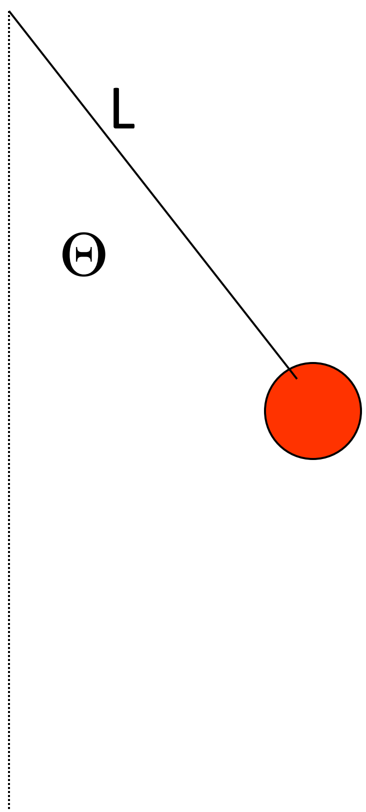
$$\Rightarrow \frac{d^2\Theta}{dt^2} = -\frac{g}{L}\Theta$$

Solution :

$$\Theta(t) = A\cos(\omega t + \varphi); \quad \omega = \sqrt{\frac{\mathrm{g}}{\mathrm{L}}}$$

Pendulum example:

Suppose L=2m, what is the period of the pendulum?



$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\mathrm{m/s}^2}{2\mathrm{m}}} = 2.2135\,\mathrm{rad/s} = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega} = 2.84\,\mathrm{s}$$

$$\Theta(t) = A\cos(\omega t + \varphi); \quad \omega = \sqrt{\frac{g}{L}}$$