PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 14:

Finish reading Chapter 6

1. **Liouville's theorem**
2. **Hamilton-Jacobi formalism**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13

Liouville's theorem:

Imagine a collection of particles obeying the Canonical equations of motion in phase space.

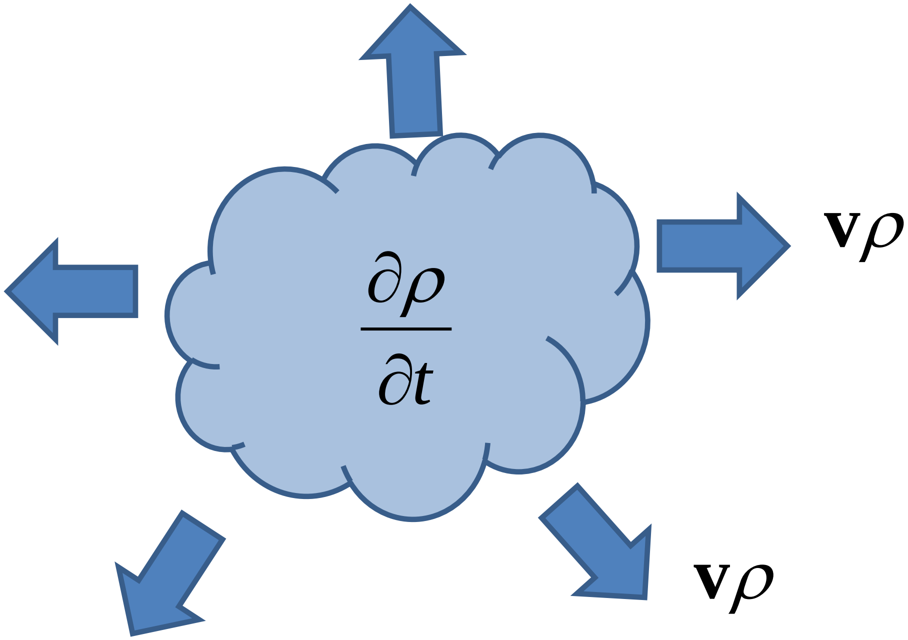
Let ρ denote the "distribution" of particles in phase space:

$$\rho = \rho\left(\{q_1 \cdots q_{3N}\},\{p_1 \cdots p_{3N}\},t\right)$$

Liouville's theorm shows that:

$$\frac{d\rho}{dt} = 0 \qquad\qquad \Rightarrow \rho \text{ is constant in time}$$

Proof of Liouville'e theorem:



Continuity equation :

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v}\rho)$$

Note : in this case, the velocity is the $6N$ dimensional vector :

$$\mathbf{v} = \left(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots \dot{\mathbf{r}}_N, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \ldots \dot{\mathbf{p}}_N\right)$$

We also have a $6N$ dimensional gradient :

$$\nabla = \left(\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}, \ldots \nabla_{\mathbf{r}_\mathrm{N}}, \nabla_{\mathbf{p}_1}, \nabla_{\mathbf{p}_2}, \ldots \nabla_{\mathbf{p}_\mathrm{N}}\right)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v}\rho)$$

$$= -\sum_{j=1}^{3N} \left[\frac{\partial}{\partial q_j}(\dot{q}_j \rho) + \frac{\partial}{\partial p_j}(\dot{p}_j \rho) \right]$$

$$= -\sum_{j=1}^{3N} \left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j \right] - \rho \sum_{j=1}^{3N} \left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right]$$

$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial^2 H}{\partial q_j \partial p_j} + \left(-\frac{\partial^2 H}{\partial p_j \partial q_j} \right) = 0$$

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j\right] - \rho \cancelto{0}{\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right]}$$

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j\right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \sum_{j=1}^{3N}\left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j\right] = \frac{d\rho}{dt} = 0$$

Notion of "Canonical" distributions

$$q_\sigma = q_\sigma\left(\{Q_1 \cdots Q_n\}, \{P_1 \cdots P_n\}, t\right) \qquad \text{for each } \sigma$$

$$p_\sigma = p_\sigma\left(\{Q_1 \cdots Q_n\}, \{P_1 \cdots P_n\}, t\right) \qquad \text{for each } \sigma$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\}, \{p_\sigma\}, t\right) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}\left(\{Q_\sigma\}, \{P_\sigma\}, t\right) + \frac{d}{dt} F\left(\{q_\sigma\}, \{Q_\sigma\}, t\right)$$

Apply Hamilton's principle:

$$\delta \int_{t_i}^{t_f} \left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}\left(\{Q_\sigma\}, \{P_\sigma\}, t\right) + \frac{d}{dt} F\left(\{q_\sigma\}, \{Q_\sigma\}, t\right) \right] dt = 0$$

$$\dot{Q}_\sigma = \frac{\partial H}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial H}{\partial Q_\sigma}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

$$\frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right) = \sum_\sigma \left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma} \right) \right) \dot{q}_\sigma - H\left(\{q_\sigma\}, \{p_\sigma\}, t \right) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma} \right) \right) \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\}, \{P_\sigma\}, t \right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma} \right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma} \right)$$

$$\Rightarrow \tilde{H}\left(\{Q_\sigma\}, \{P_\sigma\}, t \right) = H\left(\{q_\sigma\}, \{p_\sigma\}, t \right) + \frac{\partial F}{\partial t}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial H}{\partial P_\sigma} \qquad\qquad \dot{P}_\sigma = -\frac{\partial H}{\partial Q_\sigma}$$

$$\text{Suppose:}\quad \dot{Q}_\sigma = \frac{\partial H}{\partial P_\sigma} = 0 \quad \text{and} \quad \dot{P}_\sigma = -\frac{\partial H}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \qquad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose:}\quad F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t) \right)$$

$$= -\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma \left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma \right) + \frac{\partial S}{\partial t}$$

Solution :

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad\qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears :

$$\text{Assume } \{Q_\sigma\}, \{P_\sigma\}, \tilde{H} \quad \text{are constants; choose } \tilde{H} = 0$$

$$\text{Need to find } S(\{q_\sigma\}, \{P_\sigma\}, t)$$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad\qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\}, \left\{ \frac{\partial S}{\partial q_\sigma} \right\}, t \right) + \frac{\partial S}{\partial t} = 0$$

Note : S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\}, \{p_\sigma\}, t) =$$

$$\cancelto{0}{\sum_\sigma P_\sigma \dot{Q}_\sigma} - \cancelto{0}{\tilde{H}(\{Q_\sigma\}, \{P_\sigma\}, t)} + \frac{d}{dt}\left(-\cancelto{0}{\sum_\sigma P_\sigma Q_\sigma} + S(\{q_\sigma\}, \{P_\sigma\}, t) \right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$

$$\cancelto{0}{\sum_\sigma P_\sigma \dot{Q}_\sigma} - \cancelto{0}{\tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right)} + \frac{d}{dt}\left(\cancelto{0}{-\sum_\sigma P_\sigma Q_\sigma} + S\left(\{q_\sigma\},\{P_\sigma\},t\right)\right)$$

$$\int_{t_i}^{t_f}\left(\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right)\right)dt = \int_{t_i}^{t_f}\left(\frac{d}{dt}\left(S\left(\{q_\sigma\},\{P_\sigma\},t\right)\right)\right)dt$$

$$= S\left(\{q_\sigma\},\{P_\sigma\},t\right)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\text{Example:}\qquad H(\{q\},\{p\},t)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$$

$$\text{Hamilton-Jacobi Eq:}\quad H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2q^2+\frac{\partial S}{\partial t}=0$$

$$\text{Assume:}\qquad S(q,t)\equiv W(q)-Et \qquad\qquad (E\quad\text{constant})$$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

$$\text{Assume:} \qquad S(q,t) \equiv W(q) - Et \qquad\qquad (E \quad \text{constant})$$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = E$$

$$\frac{dW}{dq} = \sqrt{2mE - (m\omega)^2 q^2}$$

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

Continued:

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$

$$=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)+C$$

$$S(q,E,t)=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-Et$$

$$\frac{\partial S}{\partial E}=Q=\frac{1}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-t$$

$$\Rightarrow q(t)=\frac{\sqrt{2mE}}{m\omega}\sin(\omega(t+Q))$$