PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 14:

Finish reading Chapter 6

1. **Liouville's theorem**
2. **Hamilton-Jacobi formalism**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13

Liouville's theorem:
Imagine a collection of particles obeying the Canonical equations of motion in phase space.

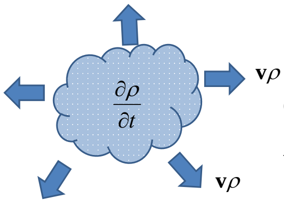
Let ρ denote the "distribution" of particles in phase space:

$$\rho = \rho(\{q_1 \cdots q_{3N}\}, \{p_1 \cdots p_{3N}\}, t)$$

Liouville's theorm shows that:

$$\frac{d\rho}{dt} = 0 \qquad \Rightarrow \rho \text{ is constant in time}$$

Proof of Liouville'e theorem:



Continuity equation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v}\rho)$$

Note: in this case, the velocity is the $6N$ dimensional vector:

$$\mathbf{v} = (\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots \dot{\mathbf{r}}_N, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \ldots \dot{\mathbf{p}}_N)$$

We also have a $6N$ dimensional gradient:

$$\nabla = (\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}, \ldots \nabla_{\mathbf{r}_N}, \nabla_{\mathbf{p}_1}, \nabla_{\mathbf{p}_2}, \ldots \nabla_{\mathbf{p}_N})$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v}\rho)$$

$$= -\sum_{j=1}^{3N} \left[\frac{\partial}{\partial q_j}(\dot{q}_j \rho) + \frac{\partial}{\partial p_j}(\dot{p}_j \rho) \right]$$

$$= -\sum_{j=1}^{3N} \left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j \right] - \rho \sum_{j=1}^{3N} \left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right]$$

$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial^2 H}{\partial q_j \partial p_j} + \left(-\frac{\partial^2 H}{\partial p_j \partial q_j} \right) = 0$$

0

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{3N} \left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j \right] - \rho \sum_{j=1}^{3N} \left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right]$$

$$\frac{\partial \rho}{\partial t} = -\sum_{j=1}^{3N} \left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j \right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \sum_{j=1}^{3N} \left[\frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j \right] = \frac{d\rho}{dt} = 0$$

Notion of "Canonical" distributions

$$q_\sigma = q_\sigma(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t) \qquad \text{for each } \sigma$$
$$p_\sigma = p_\sigma(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t) \qquad \text{for each } \sigma$$
$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

Apply Hamilton's principle:

$$\delta\int_{t_i}^{t_f}\left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)\right]dt = 0$$
$$\dot{Q}_\sigma = \frac{\partial H}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial H}{\partial Q_\sigma}$$

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Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$
$$\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t) = \sum_\sigma\left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma\right) + \frac{\partial F}{\partial t}$$
$$\sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

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$$\sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$
$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma}\right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma}\right)$$
$$\Rightarrow \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial F}{\partial t}$$

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Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial H}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial H}{\partial Q_\sigma}$$

$$\text{Suppose: } \dot{Q}_\sigma = \frac{\partial H}{\partial P_\sigma} = 0 \text{ and } \dot{P}_\sigma = -\frac{\partial H}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose: } F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

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$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$= -\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma\left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution:

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

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When the dust clears:

Assume $\{Q_\sigma\},\{P_\sigma\},\tilde{H}$ are constants; choose $\tilde{H} = 0$

Need to find $S(\{q_\sigma\},\{P_\sigma\},t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Note: S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \underbrace{P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t)}_{0} + \frac{d}{dt}\left(-\underbrace{\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

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$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \cancelto{0}{P_\sigma \dot{Q}_\sigma} - \cancelto{0}{\tilde{H}(\{Q_\sigma\}}, \{P_\sigma\},t) + \frac{d}{dt}\left(- \sum_\sigma \cancelto{0}{P_\sigma Q_\sigma} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\int_{t_i}^{t_f}\left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t)\right)dt = \int_{t_i}^{t_f}\left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t)\right)\right)dt$$

$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Example: $H(\{q\},\{p\},t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$

Hamilton-Jacobi Eq: $H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right) + \frac{\partial S}{\partial t} = 0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Assume: $S(q,t) \equiv W(q) - Et$ $\qquad$ (E constant)

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

Assume: $S(q,t) \equiv W(q) - Et$ $\qquad$ (E constant)

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + \frac{1}{2}m\omega^2 q^2 = E$$

$$\frac{dW}{dq} = \sqrt{2mE - (m\omega)^2 q^2}$$

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

Continued:

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$
$$= \frac{1}{2} q \sqrt{2mE - (m\omega)^2 q^2} + \frac{E}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) + C$$
$$S(q,E,t) = \frac{1}{2} q \sqrt{2mE - (m\omega)^2 q^2} + \frac{E}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) - Et$$
$$\frac{\partial S}{\partial E} = Q = \frac{1}{\omega} \sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}} \right) - t$$
$$\Rightarrow q(t) = \frac{\sqrt{2mE}}{m\omega} \sin(\omega(t+Q))$$