PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 15:

Start reading Chapter 4

1. **Small oscillations about equilibrium**
2. **Normal modes**

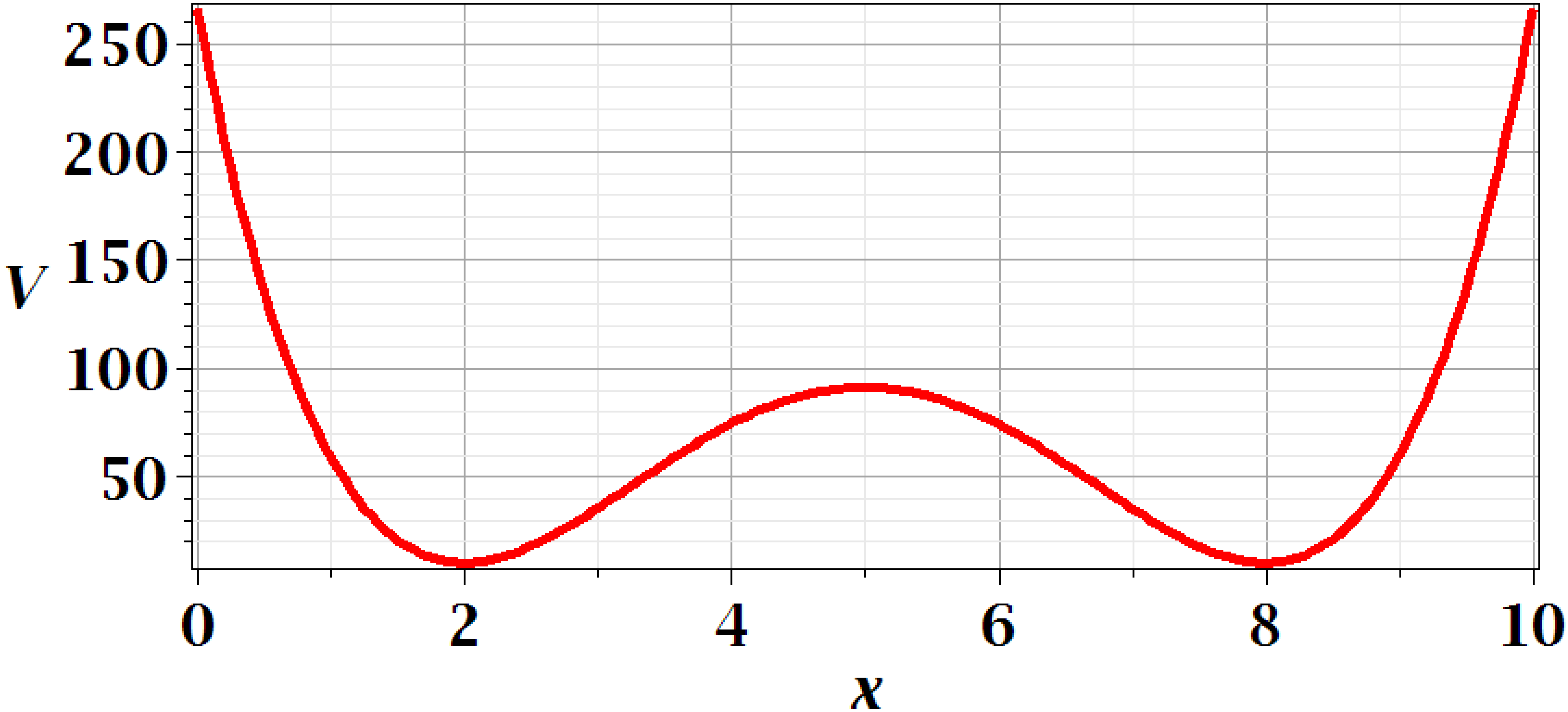
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14

Motivation for studying small oscillations – many interacting systems have stable and meta-stable configurations which are well approximated by:

$$V(x) \approx V(x_{eq}) + \tfrac{1}{2}\left(x - x_{eq}\right)^2 \left.\frac{d^2V}{dx^2}\right|_{x_{eq}} = V(x_{eq}) + \tfrac{1}{2}k\left(x - x_{eq}\right)^2$$



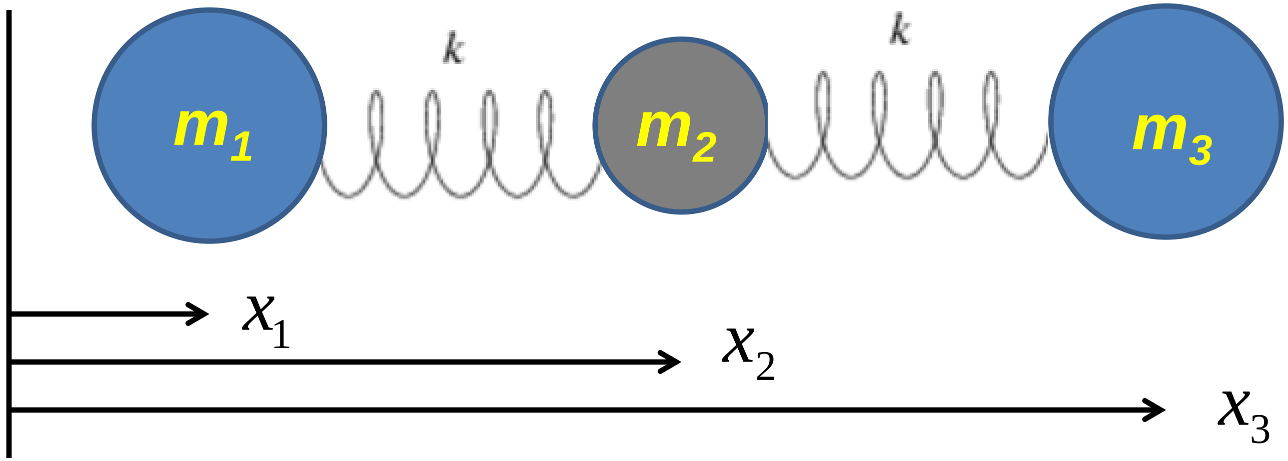
Equations of motion for a single oscillator:

$$L(x,\dot{x},t)=\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=\frac{\partial L}{\partial x} \qquad \Rightarrow m\ddot{x}=-m\omega^2 x$$

$$x(t)=A\sin(\omega t+\phi)$$

Example – linear molecule



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 - \frac{1}{2}k\left(x_2 - x_1 - \ell_{12}\right)^2 - \frac{1}{2}k\left(x_3 - x_2 - \ell_{23}\right)^2$$

$$L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2+\frac{1}{2}m_3\dot{x}_3^2$$

$$-\frac{1}{2}k\left(x_2-x_1-\ell_{12}\right)^2-\frac{1}{2}k\left(x_3-x_2-\ell_{23}\right)^2$$

Let: $\quad x_1\rightarrow x_1-x_1^0 \qquad x_2\rightarrow x_2-x_1^0-\ell_{12} \qquad x_3\rightarrow x_3-x_1^0-\ell_{12}-\ell_{23}$

$$L=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2+\frac{1}{2}m_3\dot{x}_3^2-\frac{1}{2}k\left(x_2-x_1\right)^2-\frac{1}{2}k\left(x_3-x_2\right)^2$$

Coupled equations of motion:

$$m_1\ddot{x}_1=k\left(x_2-x_1\right)$$

$$m_2\ddot{x}_2=-k\left(x_2-x_1\right)+k\left(x_3-x_2\right)=k\left(x_1-2x_2+x_3\right)$$

$$m_3\ddot{x}_3=-k\left(x_3-x_2\right)$$

Coupled equations of motion :

$$m_1\ddot{x}_1 = k\left(x_2 - x_1\right)$$

$$m_2\ddot{x}_2 = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right) = k\left(x_1 - 2x_2 + x_3\right)$$

$$m_3\ddot{x}_3 = -k\left(x_3 - x_2\right)$$

Let $x_i(t) = X_i^{\alpha} e^{-i\omega_\alpha t}$

$$-\omega_\alpha^2 m_1 X_1 = k\left(X_2^{\alpha} - X_1^{\alpha}\right)$$

$$-\omega_\alpha^2 m_2 X_2 = k\left(X_1^{\alpha} - 2X_2^{\alpha} + X_3^{\alpha}\right)$$

$$-\omega_\alpha^2 m_3 X_3 = -k\left(X_3^{\alpha} - X_2^{\alpha}\right)$$

Coupled linear equations:

$$-\omega_\alpha^2 m_1 X_1^\alpha = k\left(X_2^\alpha - X_1^\alpha\right)$$

$$-\omega_\alpha^2 m_2 X_2^\alpha = k\left(X_1^\alpha - 2X_2^\alpha + X_3^\alpha\right)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k\left(X_3^\alpha - X_2^\alpha\right)$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix}\begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

Matrix form:

$$\begin{pmatrix} k-\omega_\alpha^2 m_1 & -k & 0 \\ -k & 2k-\omega_\alpha^2 m_2 & -k \\ 0 & -k & k-\omega_\alpha^2 m_3 \end{pmatrix} \begin{pmatrix} X_1^\alpha \\ X_2^\alpha \\ X_3^\alpha \end{pmatrix} = 0$$

More convenient form:

Let $Y_i \equiv \sqrt{m_i}\, X_i$ Equations for Y_i take the form:

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

where $\kappa_{ij} = \kappa_{ji} \equiv \dfrac{k}{\sqrt{m_i m_j}}$

Digression:

$$\text{Eigenvalue properties of matrices} \qquad \mathbf{M}\mathbf{y}_\alpha = \lambda_\alpha \mathbf{y}_\alpha$$

$$\text{Hermitian matrix:} \qquad H_{ij} = H^*_{ji}$$

Theorem for Hermitian matrices:

$$\lambda_\alpha \ \text{ have real values and } \ \mathbf{y}^*_\alpha \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

$$\text{Unitary matrix:} \qquad \mathbf{U}\mathbf{U}^* = \mathbf{I}$$

$$\left|\lambda_\alpha\right| = 1 \ \text{ and } \ \mathbf{y}^*_\alpha \cdot \mathbf{y}_\beta = \delta_{\alpha\beta}$$

In our case:

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} & 0 \\ -\kappa_{12} & 2\kappa_{22} & -\kappa_{23} \\ 0 & -\kappa_{23} & \kappa_{33} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

for $m_1 = m_3 \equiv m_O$ and $m_2 \equiv m_C$ $\quad (\mathrm{CO_2})$

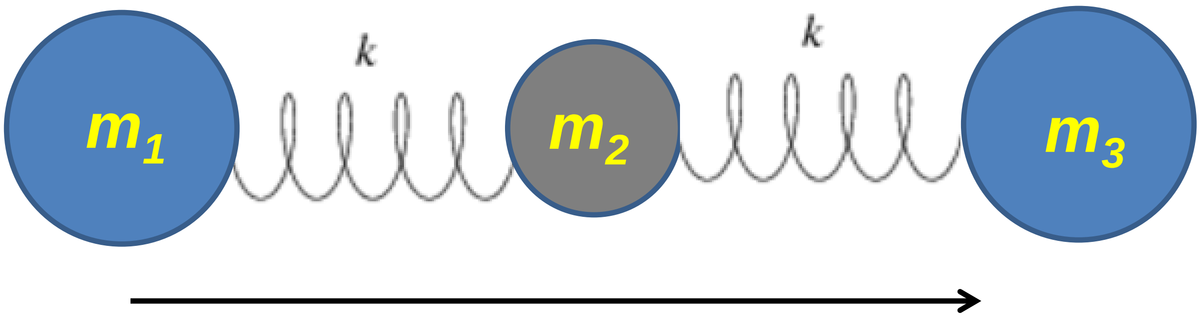
$$\begin{pmatrix} \kappa_{OO} & -\kappa_{OC} & 0 \\ -\kappa_{OC} & 2\kappa_{CC} & -\kappa_{OC} \\ 0 & -\kappa_{OC} & \kappa_{OO} \end{pmatrix} \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix} = \omega_\alpha^2 \begin{pmatrix} Y_1^\alpha \\ Y_2^\alpha \\ Y_3^\alpha \end{pmatrix}$$

Eigenvalues and eigenvectors :

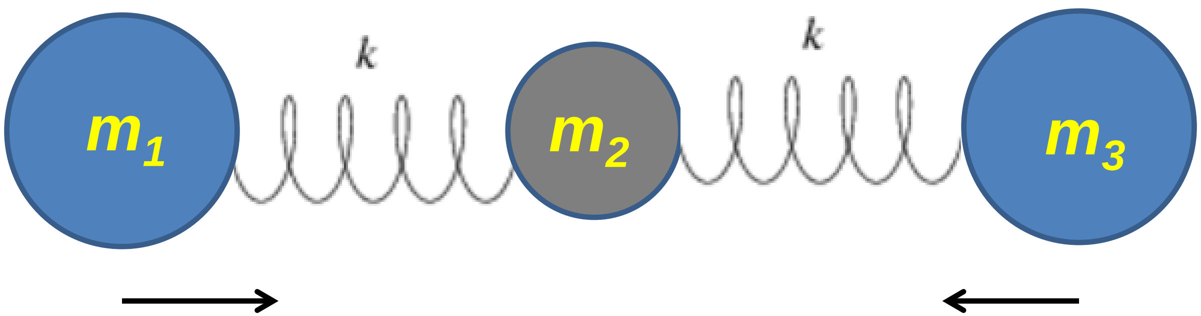
$$\omega_1^2 = 0 \qquad \begin{pmatrix} Y_1^1 \\ Y_2^1 \\ Y_3^1 \end{pmatrix} = N_1 \begin{pmatrix} \sqrt{\frac{m_O}{m_C}} \\ 1 \\ \sqrt{\frac{m_O}{m_C}} \end{pmatrix}, \quad \begin{pmatrix} X_1^1 \\ X_2^1 \\ X_3^1 \end{pmatrix} = N'_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m_O} \qquad \begin{pmatrix} Y_1^2 \\ Y_2^2 \\ Y_3^2 \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} X_1^2 \\ X_2^2 \\ X_3^2 \end{pmatrix} = N'_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

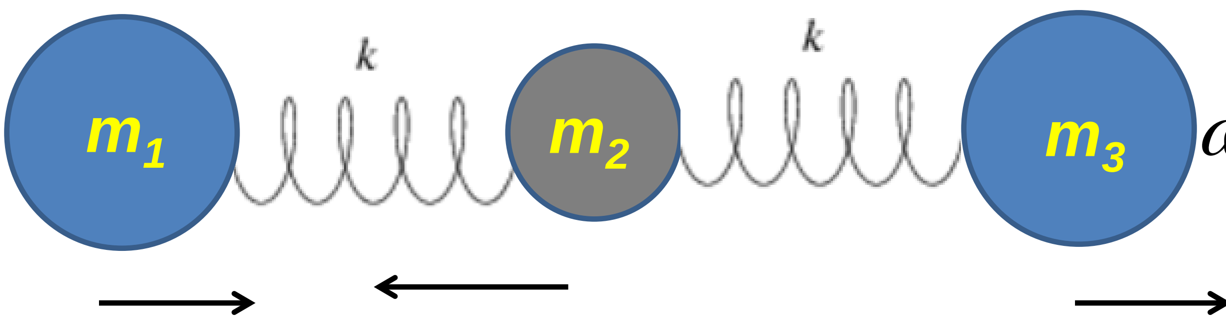
$$\omega_3^2 = \frac{k}{m_O} + \frac{2k}{m_C} \qquad \begin{pmatrix} Y_1^3 \\ Y_2^3 \\ Y_3^3 \end{pmatrix} = N_3 \begin{pmatrix} 1 \\ -2\sqrt{\frac{m_O}{m_C}} \\ 1 \end{pmatrix}, \quad \begin{pmatrix} X_1^3 \\ X_2^3 \\ X_3^3 \end{pmatrix} = N'_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$



$$\omega_1 = 0$$



$$\omega_2 = \sqrt{\frac{k}{m_O}}$$



$$\omega_3 = \sqrt{\frac{k}{m_O} + \frac{2k}{m_C}}$$

General solution :

$$x_i(t) = \Re\left(\sum_\alpha C^\alpha X_i^\alpha e^{-i\omega_\alpha t} \right)$$

For example, normal mode amplitudes

C^α can be determined from initial conditions