PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 21:

More mathematical methods

1. Laplace transforms
2. Fourier transforms
3. FFT

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/29/2012	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/31/2012	Chap. 1	Scattering theory continued	#2
3	Mon, 9/03/2012	Chap. 1	Scattering theory continued	#3
4	Wed, 9/05/2012	Chap. 1 & 2	Scattering theory/Accelerated coordinate frame	#4
5	Fri, 9/07/2012	Chap. 2	Accelerated coordinate frame	#5
6	Mon, 9/10/2012	Chap. 3	Calculus of Variation	#6
7	Wed, 9/12/2012	Chap. 3	Calculus of Variation continued	
8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px}\phi(x)dx. \quad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \quad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \quad (26)$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \le x \le \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p + \gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma + p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma + p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - e^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

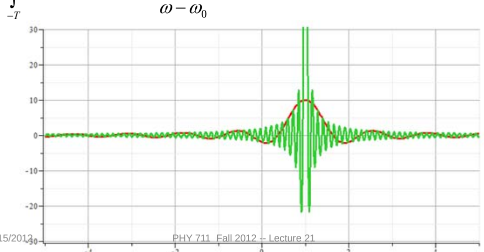
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega - \omega_0)t} = 2\pi\delta(\omega - \omega_0)$$

Note that

$$\int_{-T}^{T} dt\, e^{-i(\omega - \omega_0)t} = \frac{2\sin[(\omega - \omega_0)T]}{\omega - \omega_0}$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t)=\int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Check:

$$f(t)=\int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'}\right) e^{-i\omega t}$$

$$f(t)=\int_{-\infty}^{\infty} dt'\, f(t') \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{i\omega(t'-t)}\right)=\int_{-\infty}^{\infty} dt'\, f(t')\ \delta(t'-t)$$

Properties of Fourier transforms - - Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\, \left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} d\omega \left(F(\omega)\right)^* F(\omega)$$

Fourier transform for periodic function:

Suppose $f(t+nT)=f(t)$ for and integer n

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{n=-\infty}^{\infty}\left(\int_0^T dt\, f(t) e^{i\omega(t+nT)}\right)$$

Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where } \Omega\equiv\frac{2\pi}{T}$$

Some details:

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right)=2\pi\sum_{\nu}\delta(\omega T-\nu\Omega T)=\frac{2\pi}{T}\sum_{\nu}\delta(\omega-\nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where } \Omega\equiv\frac{2\pi}{T}$$

$$\Rightarrow F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty}\Omega\,\delta(\omega-\nu\Omega)\left(\int_0^T dt\, f(t) e^{i\omega t}\right)$$

Thus, for a periodic function

$$f(t)=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)|\leq\varepsilon \quad \text{for} \quad |\nu|\geq N$$

Define a periodic transform function function

$$\tilde{F}(\nu\Omega)\equiv\tilde{F}(\nu\Omega+\nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t)=\sum_{\nu=-\infty}^{\infty}\tilde{F}(\nu\Omega)e^{-i\nu\Omega t} \;=\frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\tilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

Doubly periodic functions

$$t\rightarrow\frac{\mu T}{2N+1}$$

$$\tilde{f}_{\mu}=\frac{1}{2N+1}\sum_{\nu=-N}^{N}\tilde{F}_{\nu}e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_{\nu}=\sum_{\mu=-N}^{N}\tilde{f}_{\mu}e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N+1\rightarrow M$$

$$\tilde{f}_{\mu}=\frac{1}{M}\sum_{\nu=0}^{M-1}\tilde{F}_{\nu}e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_{\nu}=\sum_{\mu=0}^{M}\tilde{f}_{\mu}e^{i2\pi\nu\mu/M}$$

Note that for $W=e^{i2\pi/M}$

$$\tilde{F}_0=\tilde{f}_0W^0+\tilde{f}_1W^0+\tilde{f}_2W^0+\tilde{f}_3W^0+\cdots$$

$$\tilde{F}_1=\tilde{f}_0W^0+\tilde{f}_1W^1+\tilde{f}_2W^2+\tilde{f}_3W^3+\cdots$$

$$\tilde{F}_2=\tilde{f}_0W^0+\tilde{f}_1W^2+\tilde{f}_2W^4+\tilde{f}_3W^6+\cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)