PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

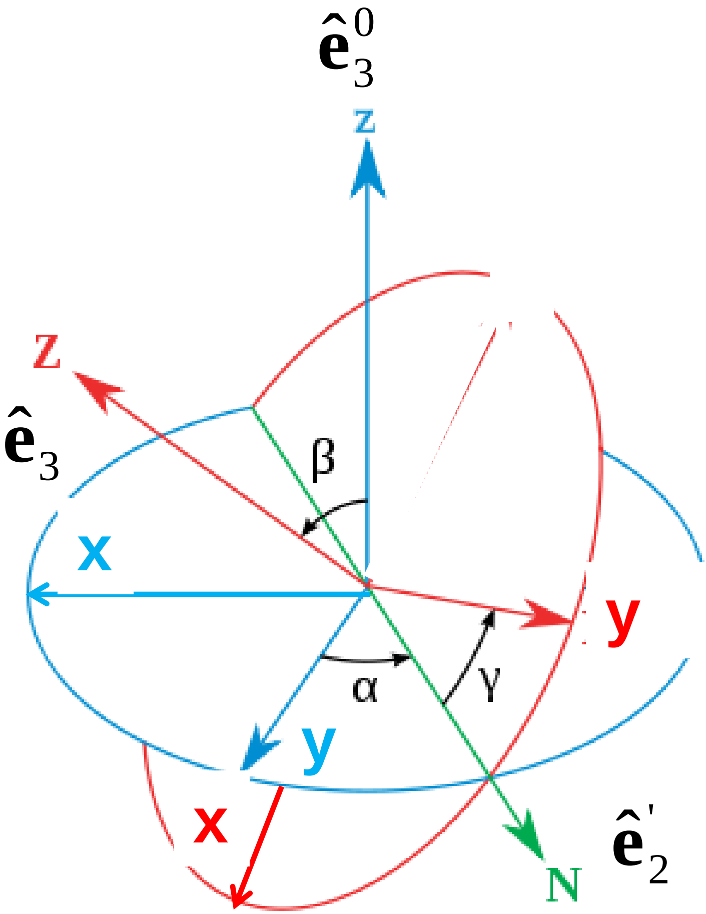
Plan for Lecture 25:

Rigid body rotational motion (Chap. 5)

1. **Motion of a symmetric top**

11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} = &\left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ &+ \left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ &+ \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

Rotational kinetic energy

$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\tilde{\omega}_1^2+\frac{1}{2}I_2\tilde{\omega}_2^2+\frac{1}{2}I_3\tilde{\omega}_3^2$$

$$=\frac{1}{2}I_1\left[\dot{\alpha}\left(-\sin\beta\cos\gamma\right)+\dot{\beta}\sin\gamma\right]^2$$

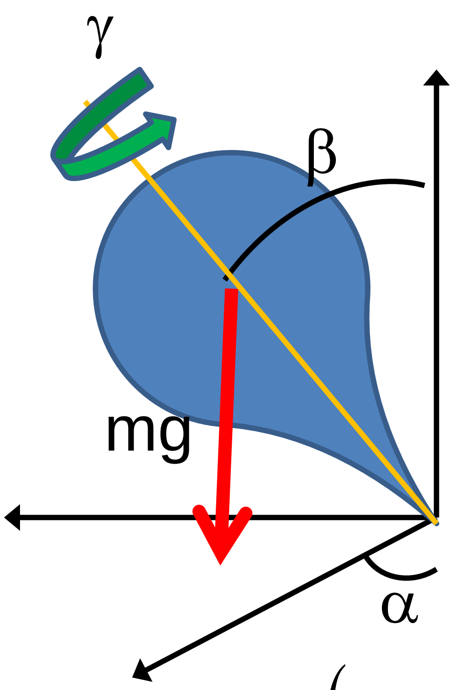
$$+\frac{1}{2}I_2\left[\dot{\alpha}\left(\sin\beta\sin\gamma\right)+\dot{\beta}\cos\gamma\right]^2$$

$$+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2$$

If $\quad I_1 = I_2$:

$$T\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2$$

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) +$$

$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]^2 - Mgl\cos\beta$$

$$L\left(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}\right)=\frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+$$

$$\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2-Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha=\frac{\partial L}{\partial\dot{\alpha}}=I_1\dot{\alpha}\sin^2\beta+I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]\cos\beta$$

$$p_\gamma=\frac{\partial L}{\partial\dot{\gamma}}=I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]$$

$$E=\frac{1}{2}I_1\dot{\beta}^2+\frac{p_\gamma^2}{2I_3}+V_{eff}\left(\beta\right)$$

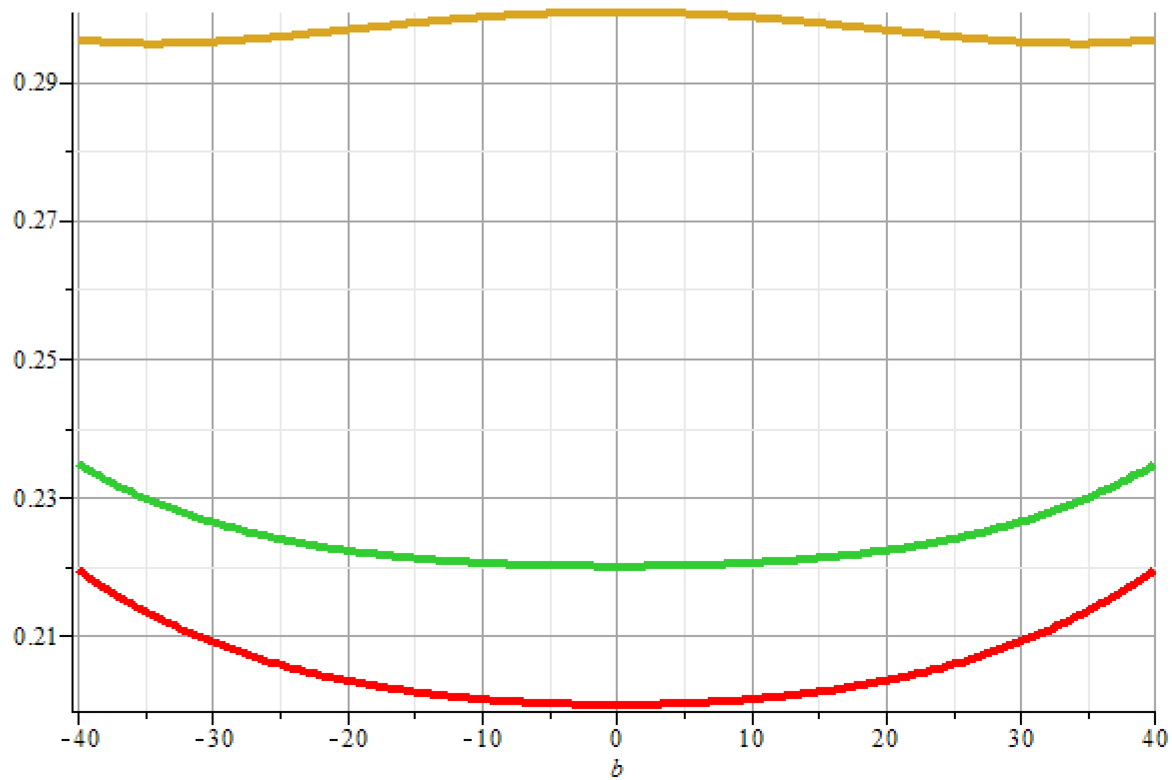
$$L\left(\beta,\dot{\beta}\right)=\frac{1}{2}I_1\dot{\beta}^2+\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+\frac{p_\gamma^2}{2I_3}-Mgl\cos\beta$$

$$V_{eff}\left(\beta\right)=\frac{\left(p_\alpha-p_\gamma\cos\beta\right)^2}{2I_1\sin^2\beta}+Mgl\cos\beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{\left(p_\alpha - p_\gamma \cos\beta\right)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma \cos\beta\right)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

Stable/unstable solutions near β=0



Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{\left(p_\alpha - p_\gamma \cos\beta\right)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^{\ 2}}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \frac{1}{2}\beta^2\right)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma^{\ 2}}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

$\Rightarrow$ Stable solution if

$$\mathrm{p}_\gamma \geq \sqrt{4MglI_1}$$

More general case: